Discontinuous Percolation

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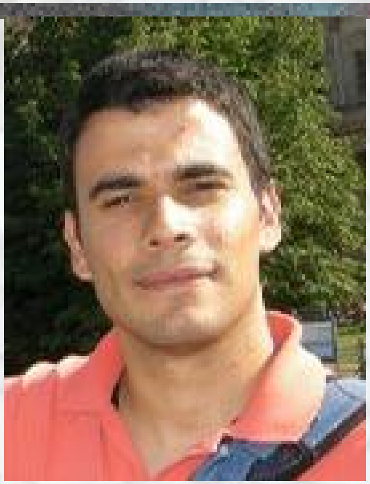
Collaborators



Nuno Araújo

José Soares Andrade

Bob Ziff

Julian Schrenk

Young-Sul Cho

Byungnam Kahng

Lucas de Oliveira

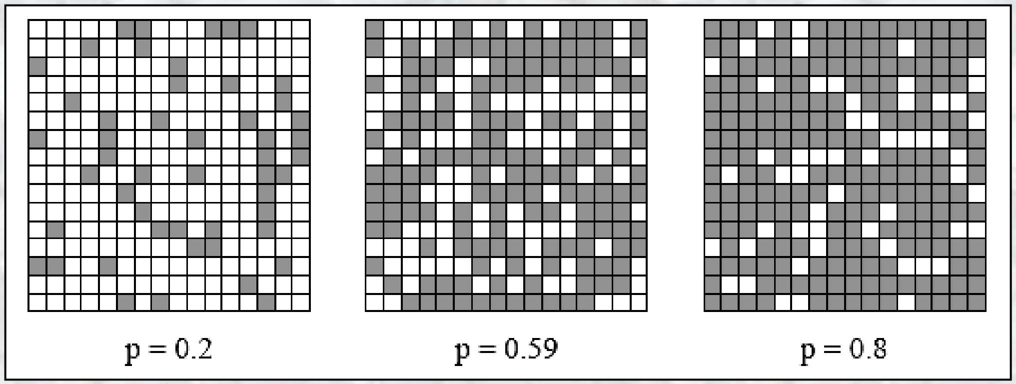
Peter Grassberger

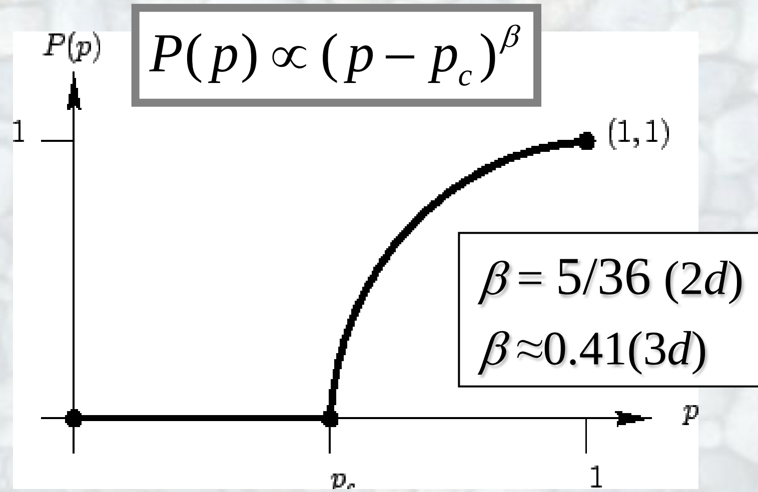
Classical Percolation

Site percolation on the square lattice:

$P(p)$ = fraction of sites in the largest cluster

Occupy randomly sites with probability p.





Neighboring occupied sites are „connected“ and belong to the same cluster.

Above a critical theshold p_c one has a spanning cluster.

The phase transition is continuous (of second order) with universal critical exponents.

Quest for First Order Transition

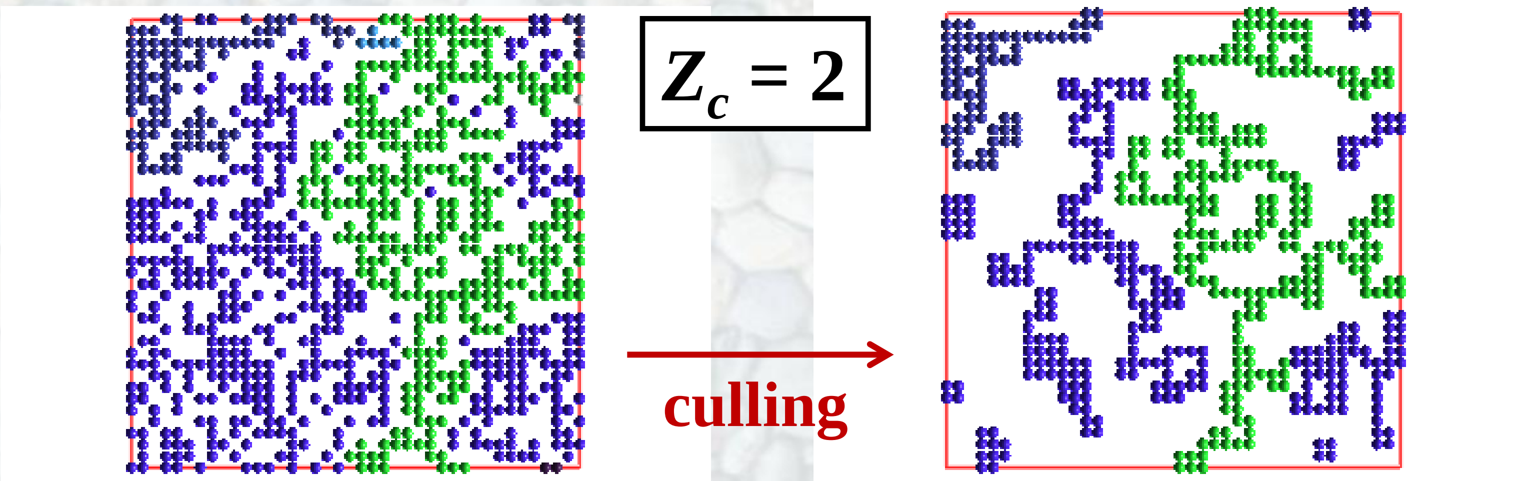
Breaking of a dam

Volcano eruption

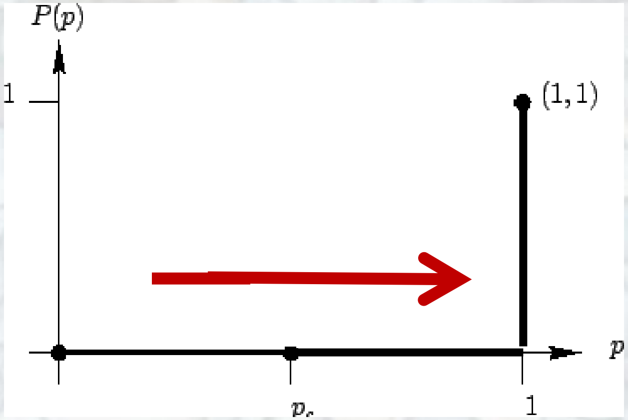
Financial bubble

First Order Transition in Percolation

Bootstrap Percolation



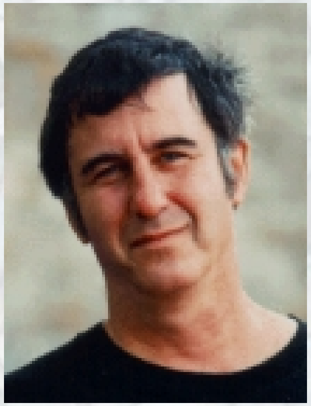
The transition is first order (at $p_c = 1$) on simple cubic and triangular lattice when $Z_c \geq 4$ and on square lattice when $Z_c \geq 3$.



The Saga of Explosive Percolation

Dimitris Achlioptas

Raissa D'Souza



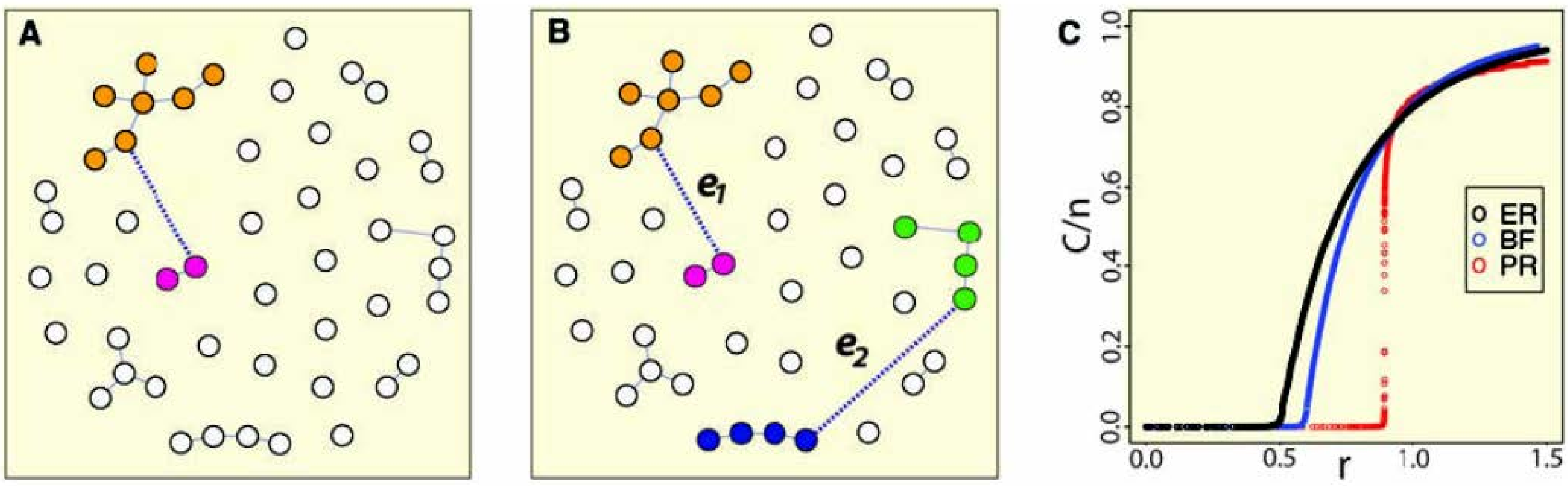
Joel Spencer

D. Achlioptas, R. M. D'Souza and J. Spencer, Science 323, 1453 (2009)

Product Rule (PR)

- **Consider a fully connected graph.**
- **Select randomly two bonds and occupy the one which creates the smaller cluster.**

classical percolation **product rule**



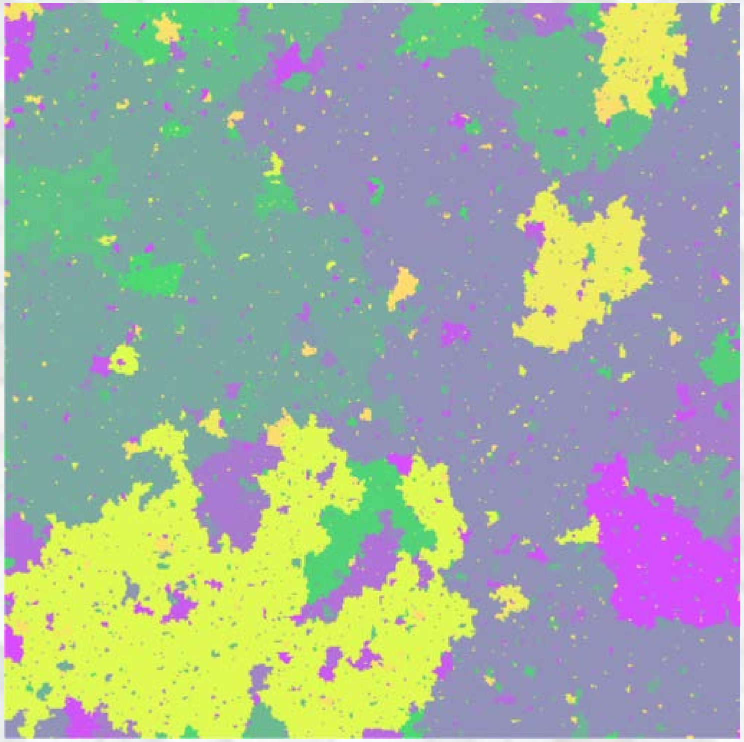
D. Achlioptas, R. M. D'Souza and J. Spencer, Science 323, 1453 (2009)

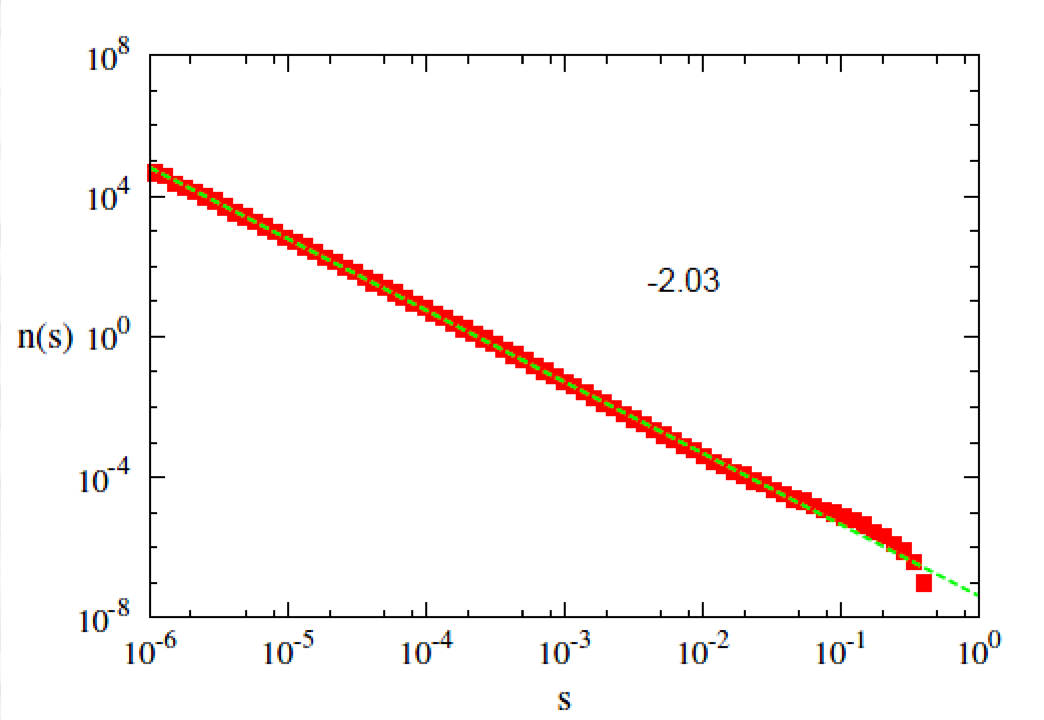
Product Rule (PR)

cluster size distribution n_s

on the square lattice:

$$n_s \propto s^{-\tau}$$





Y. S. Cho et al., Phys. Rev. E 82, 042102 (2010)

However, ...

Transition continuous in thermodynamic limit

J. Nagler, A. Levina and T. Timme, Nature Phys. 7, 2645 (2010)

O. Riordan and L. Warnke, Science, 333, 322 (2011)

R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett., 105, 255701 (2010)

But what happens in finite dimension ??

Best-of-*m* Model

José Soares Andrade Jr.

- Select randomly *m* bonds and occupy the one which creates the smaller cluster

This is a straightforward generalization of the Product Rule which corresponds to $m = 2$.
$m = 1$ is classical percolation.

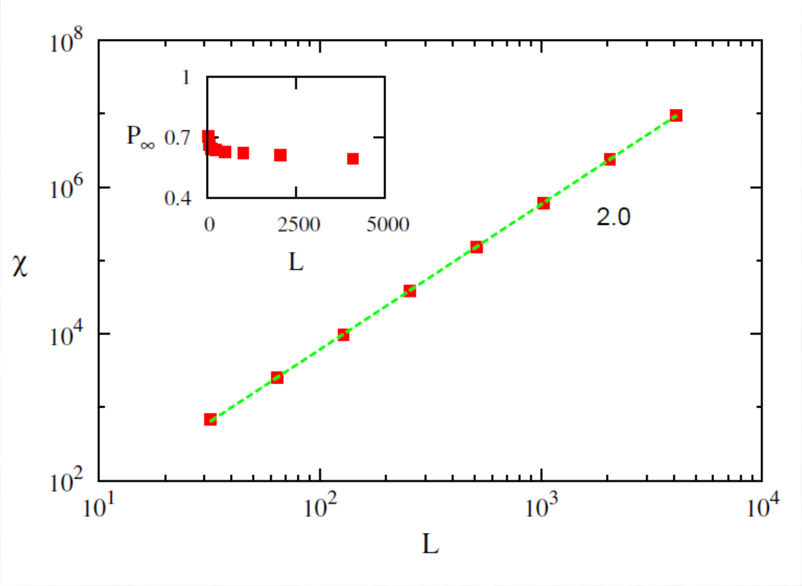
Best-of-*m* Model

$$\chi = {\sum_i}' s_i^2$$

$$P_\infty = s_{\max} / N$$

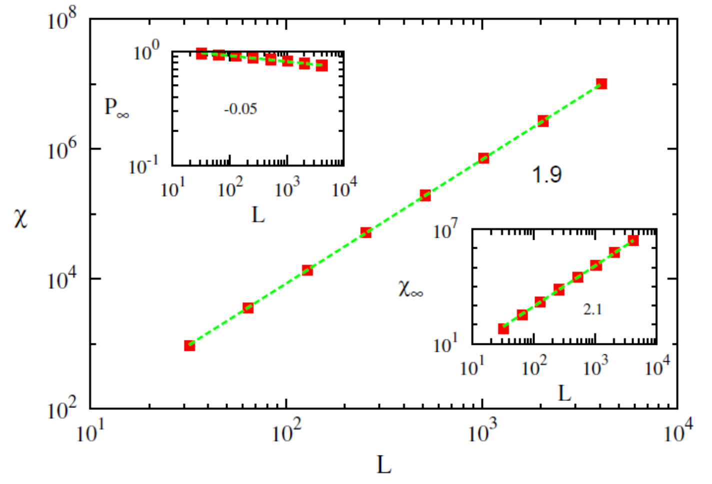
$$\chi_\infty = \sqrt{\langle s_{\max}^2 \rangle - \langle s_{\max} \rangle^2}$$

***m* = 10**

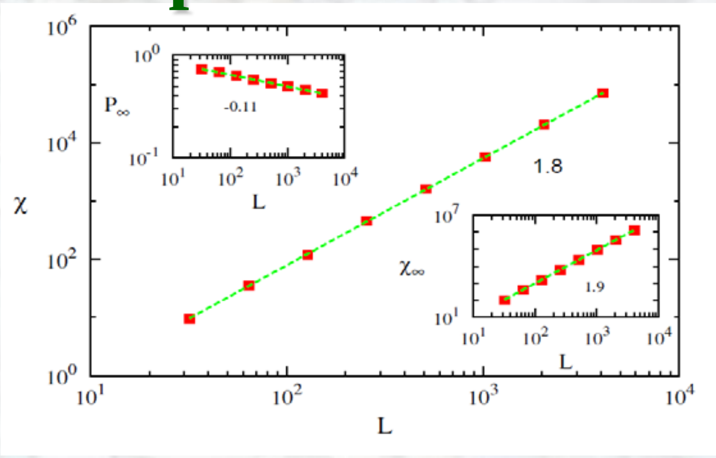


at p_c on square lattice

***m* = 2**



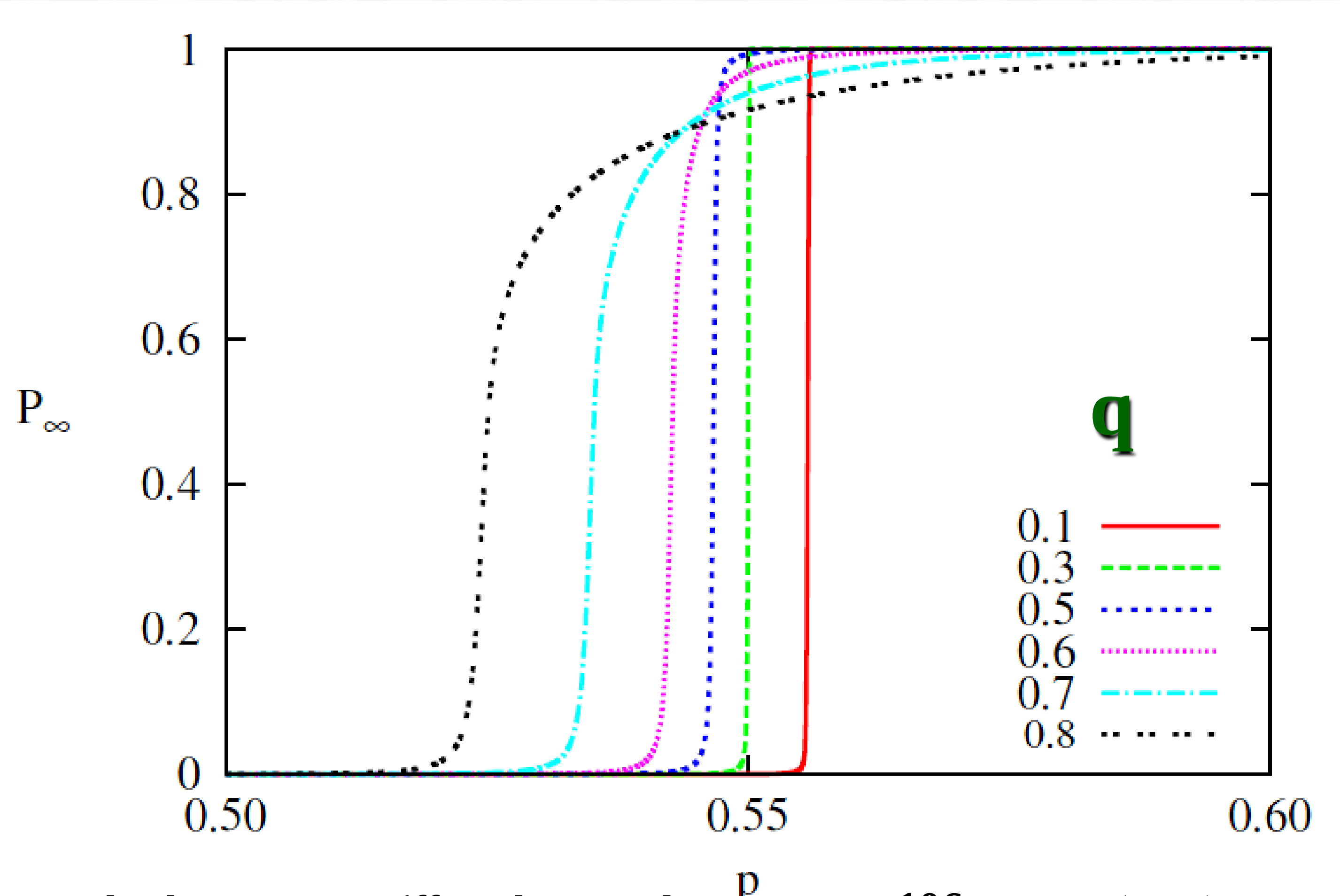
classical percolation



Mixing m = 10 with m = 1

q is the fraction of m = 1 bonds

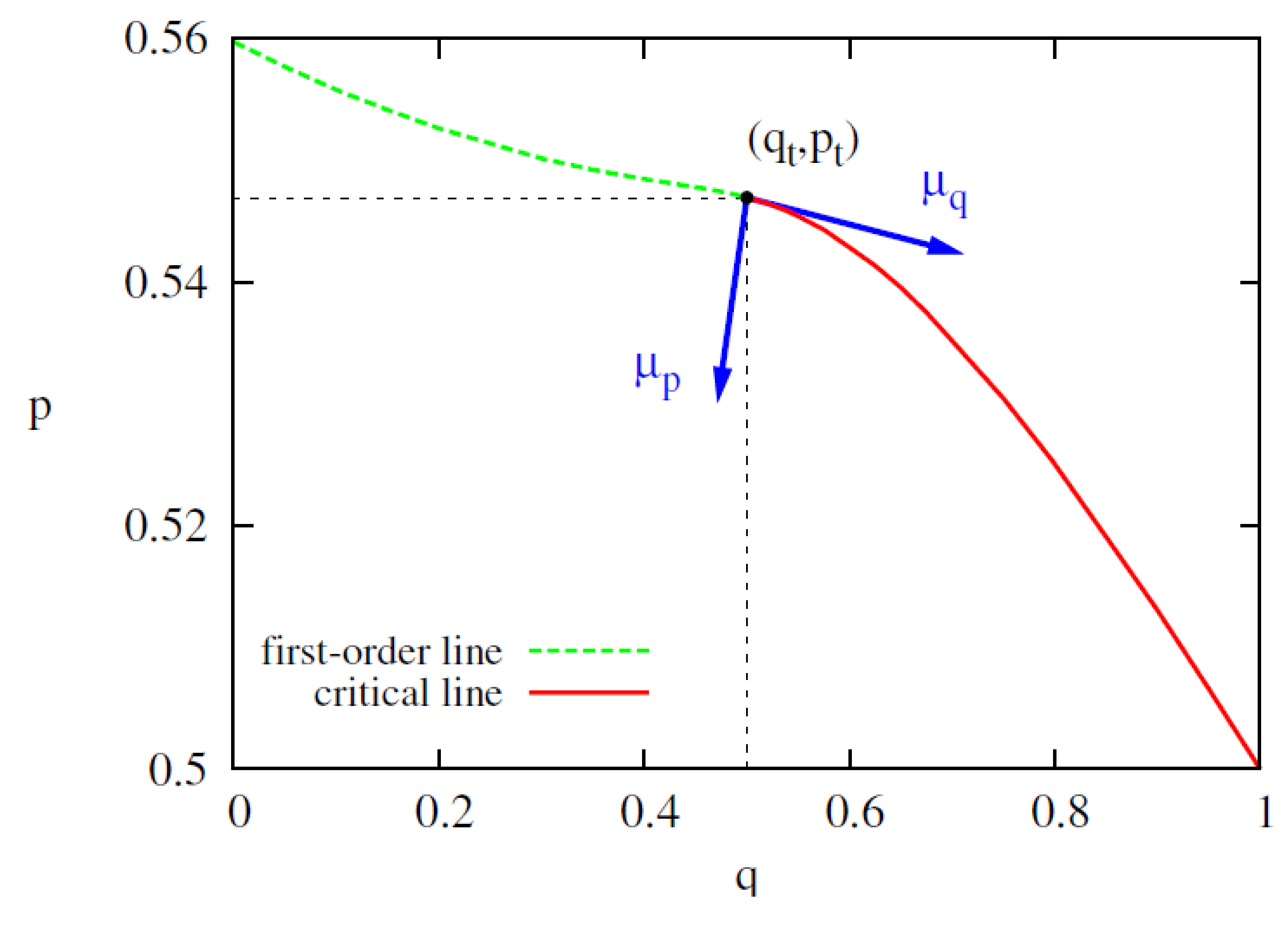
Bob Ziff



N. A. M. Araújo, J. S. Andrade Jr., R. M. Ziff, and HJH, Phys.Rev.Lett. 106, 095703 (2011)

Mixing m = 10 with m = 1

tricritical point

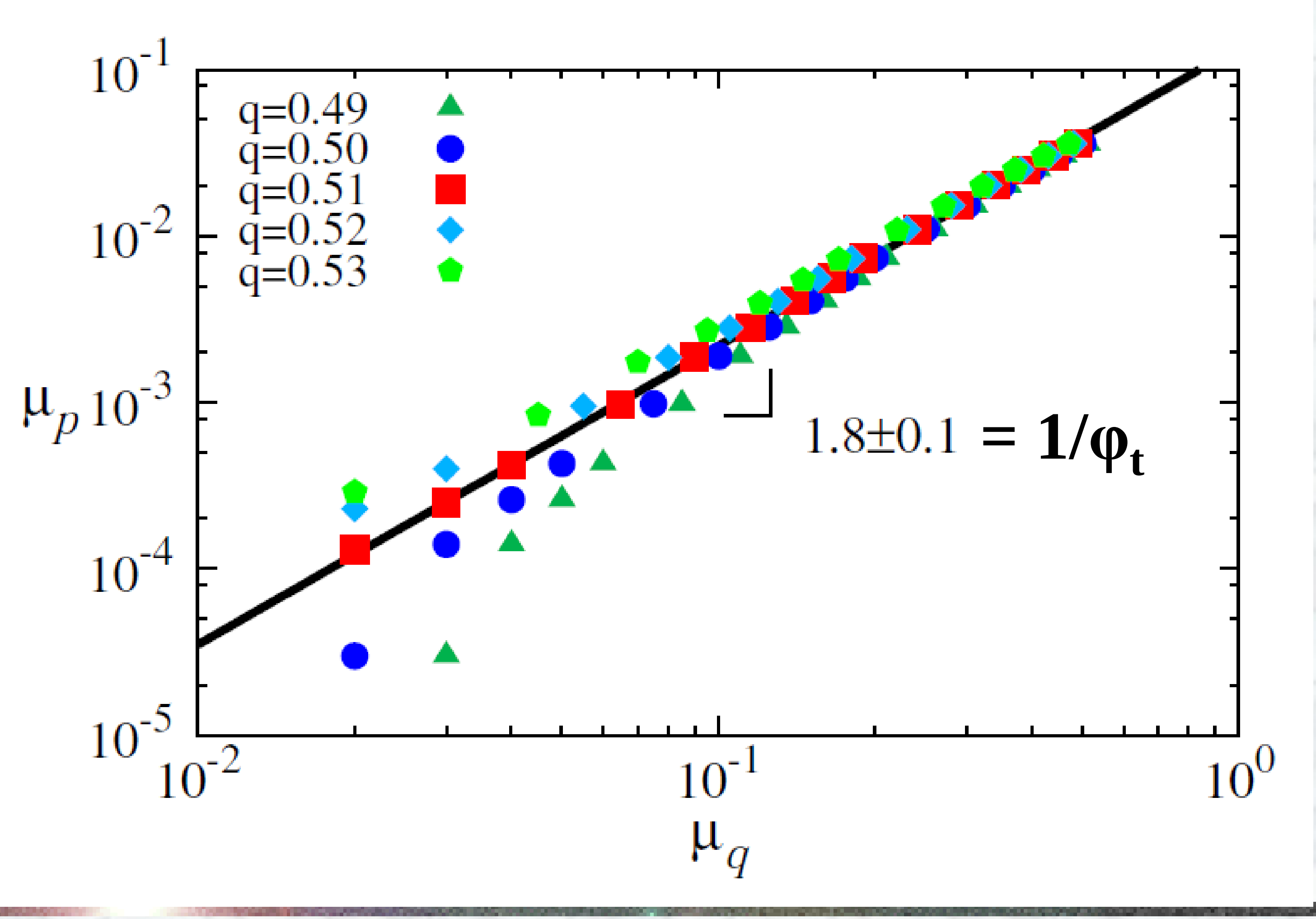


N. A. M. Araújo, J. S. Andrade Jr., R. M. Ziff, and HJH, Phys.Rev.Lett. 106, 095703 (2011)

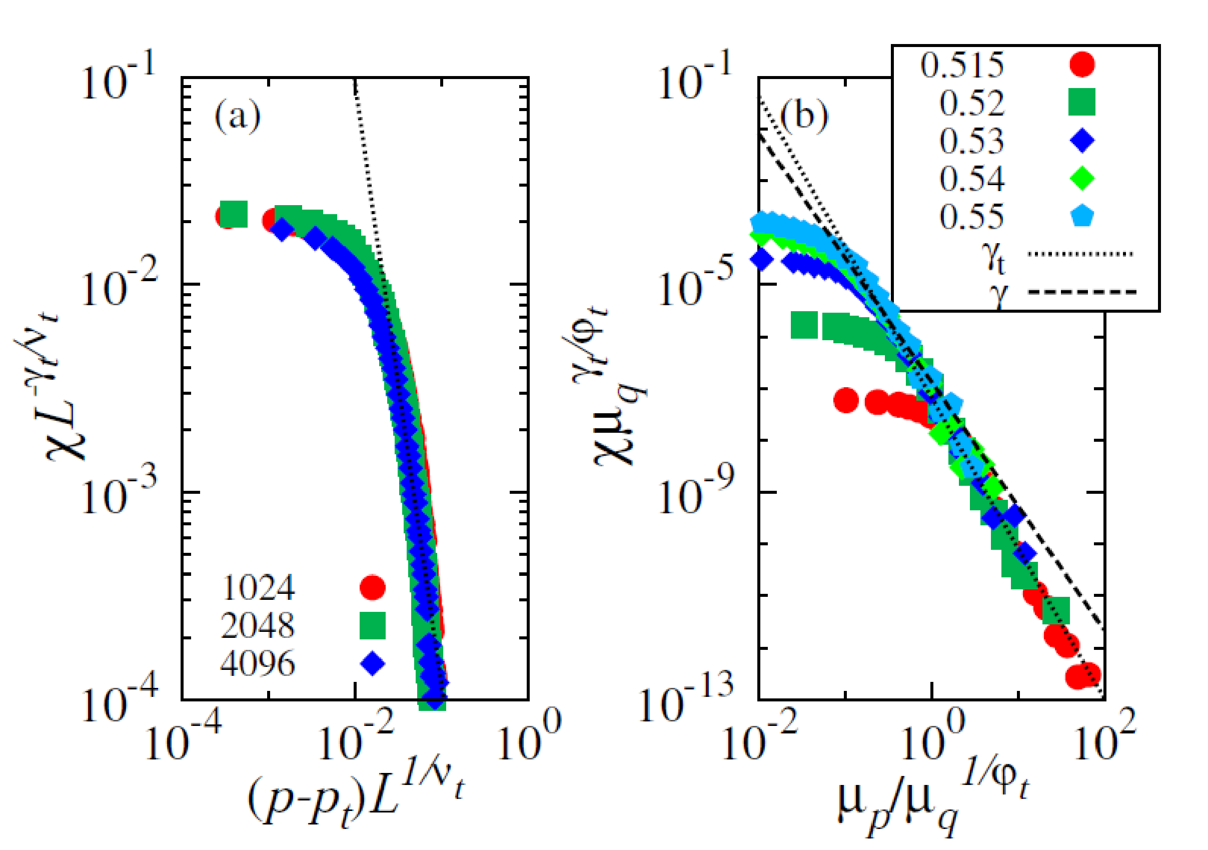
Mixing m = 10 with m = 1

tricritical scaling

$$\mu_p \propto \mu_q^{\frac{1}{\varphi_t}}$$



Mixing m = 10 with m = 1

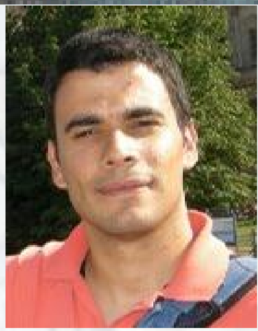


tricritical scaling

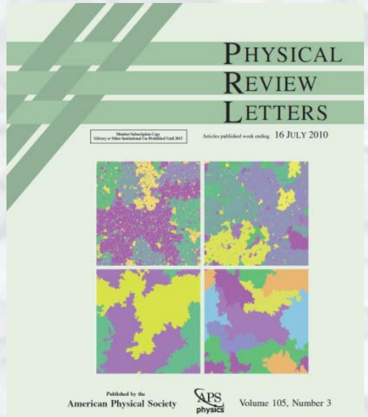
$$\chi = L^{\frac{\gamma_t}{\nu_t}} \mathcal{F}[(p-p_c)L^{\frac{1}{\nu_t}}]$$

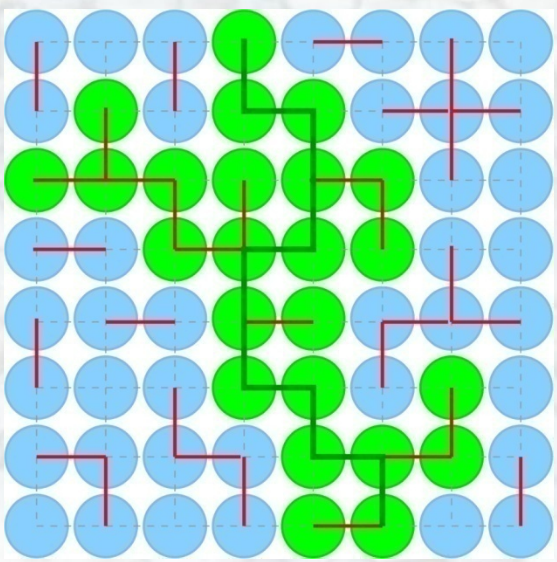
$$\chi\left(\mu_p,\mu_q\right) = \mu_q^{-\frac{\gamma_t}{\varphi_t}} \mathcal{F}_{cross}[\mu_p \mu_q^{-\frac{1}{\varphi_t}}]$$

Largest Cluster Model



Nuno Araújo





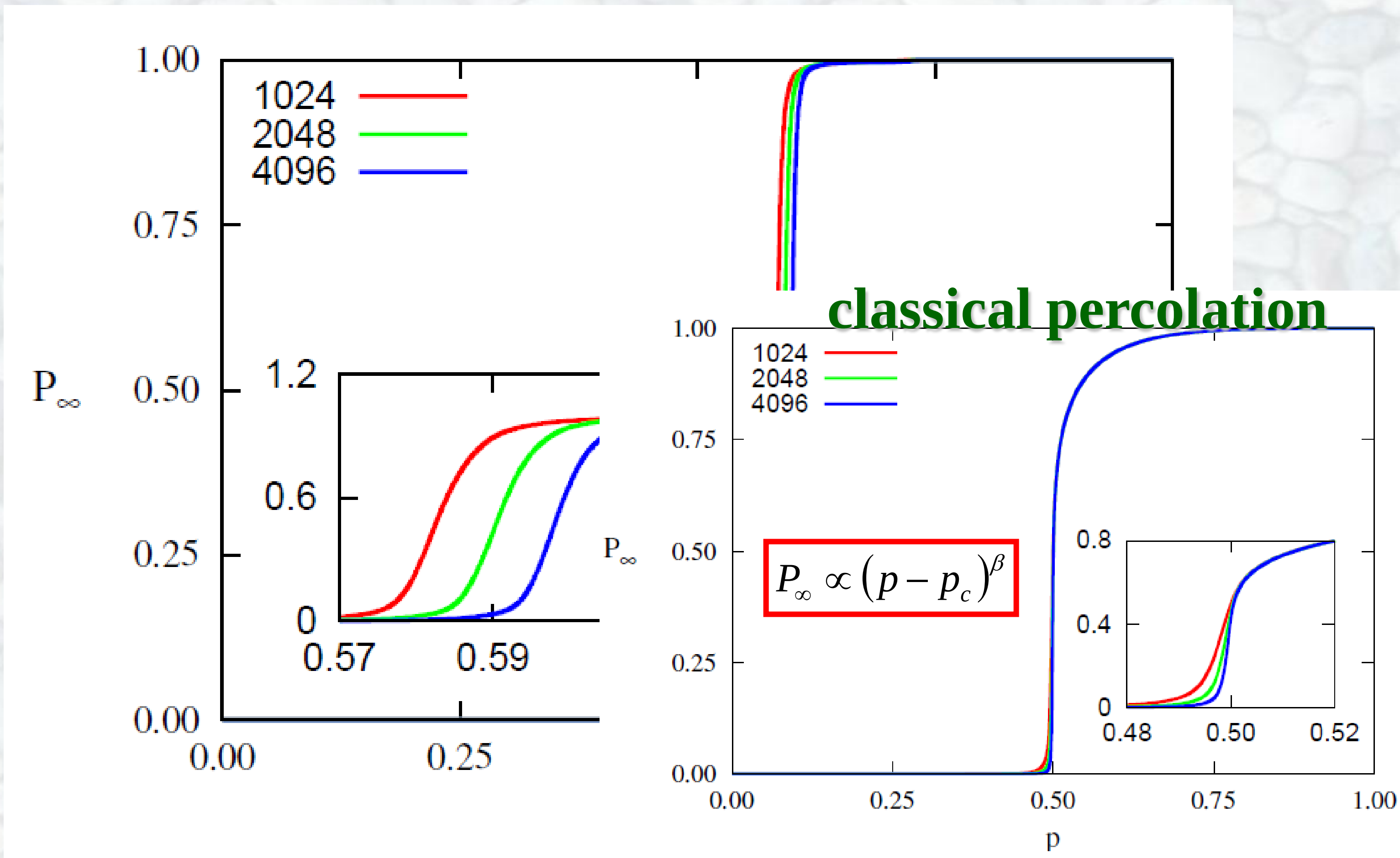
- select randomly a bond
- if not related with the largest cluster occupy it
- else, occupy it with probability

$$q = \exp\left[-\left(\frac{s - \bar{s}}{\bar{s}} \right)^2 \right]$$

Nuno Araújo and HJH, *Phys. Rev. Lett.* 105, 035701 (2010)

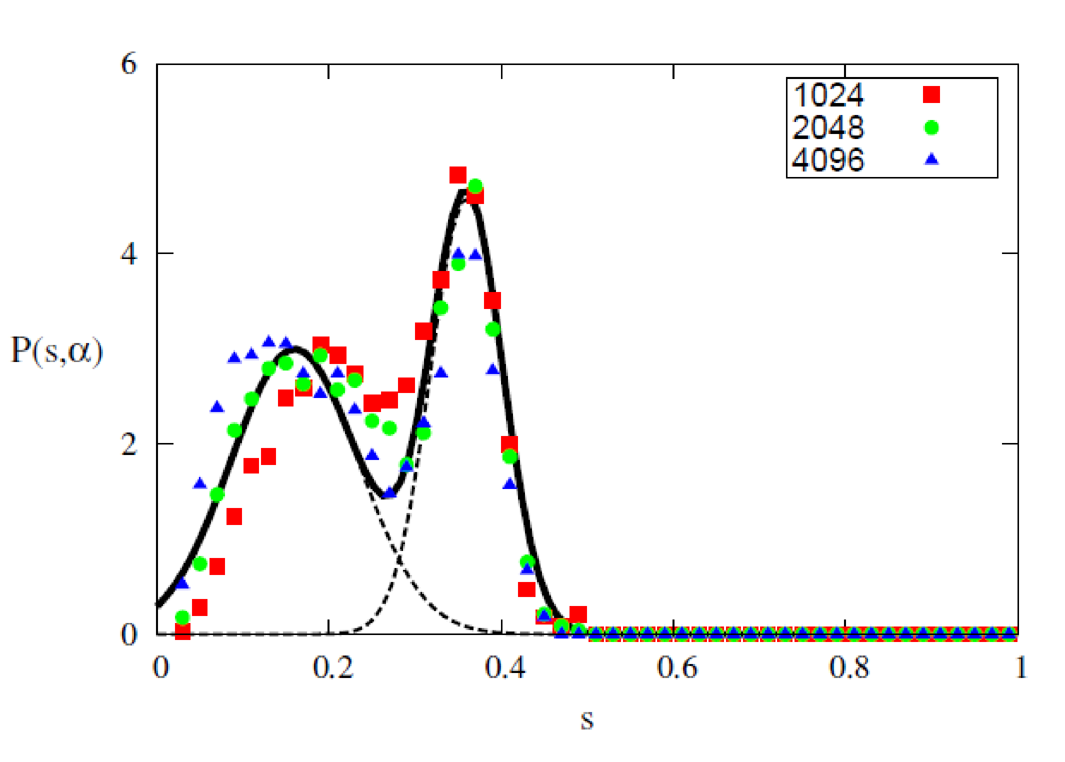
Largest Cluster Model

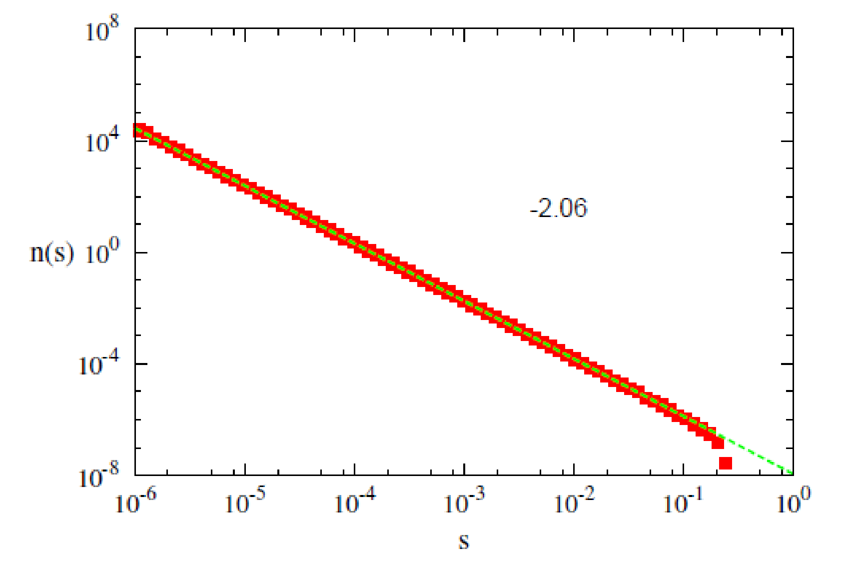
order parameter: P_∞ = fraction of sites in largest cluster



Largest Cluster Model

at p_c — cluster size distribution

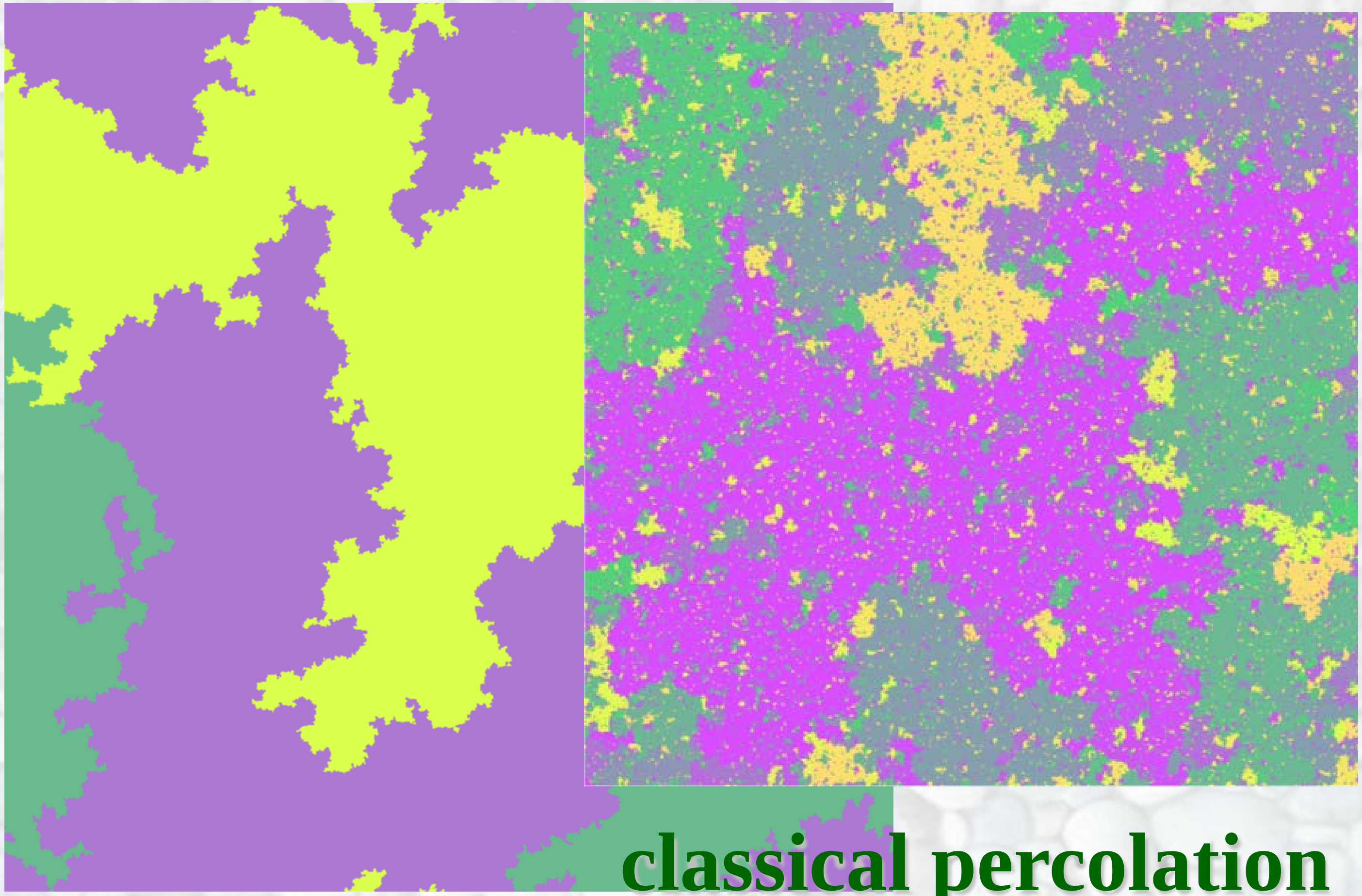




classical percolation

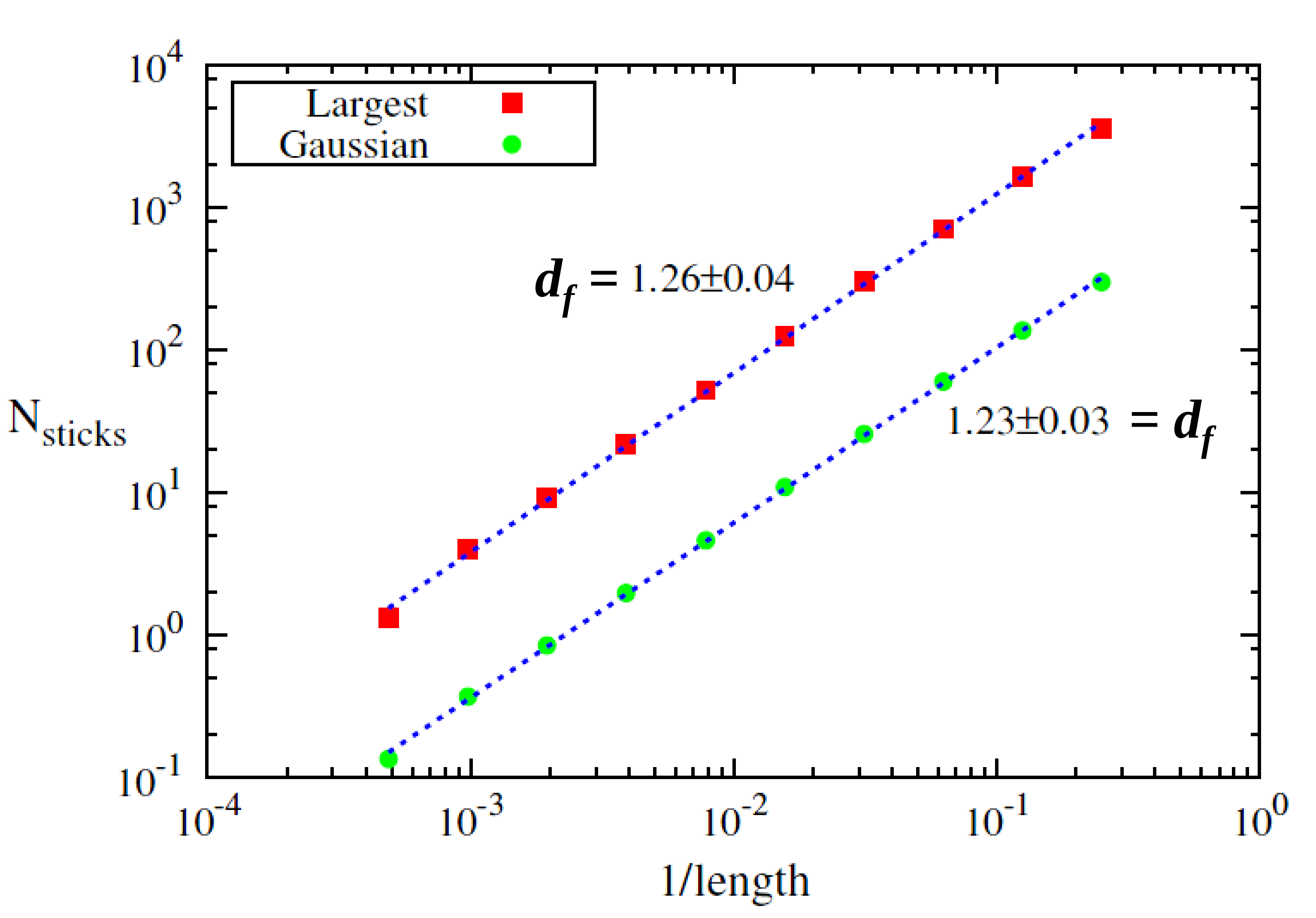
Largest Cluster Model

at p_c



classical percolation

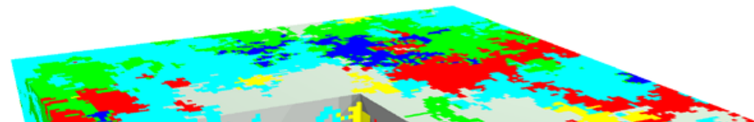
Surface of the clusters

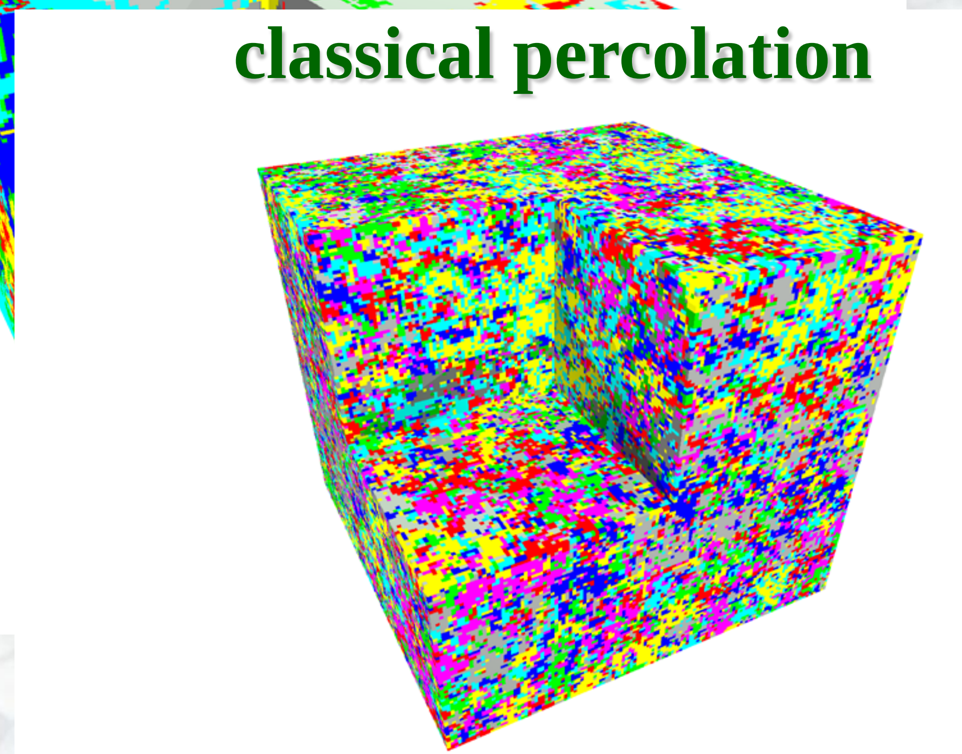


Largest cluster Model in 3D

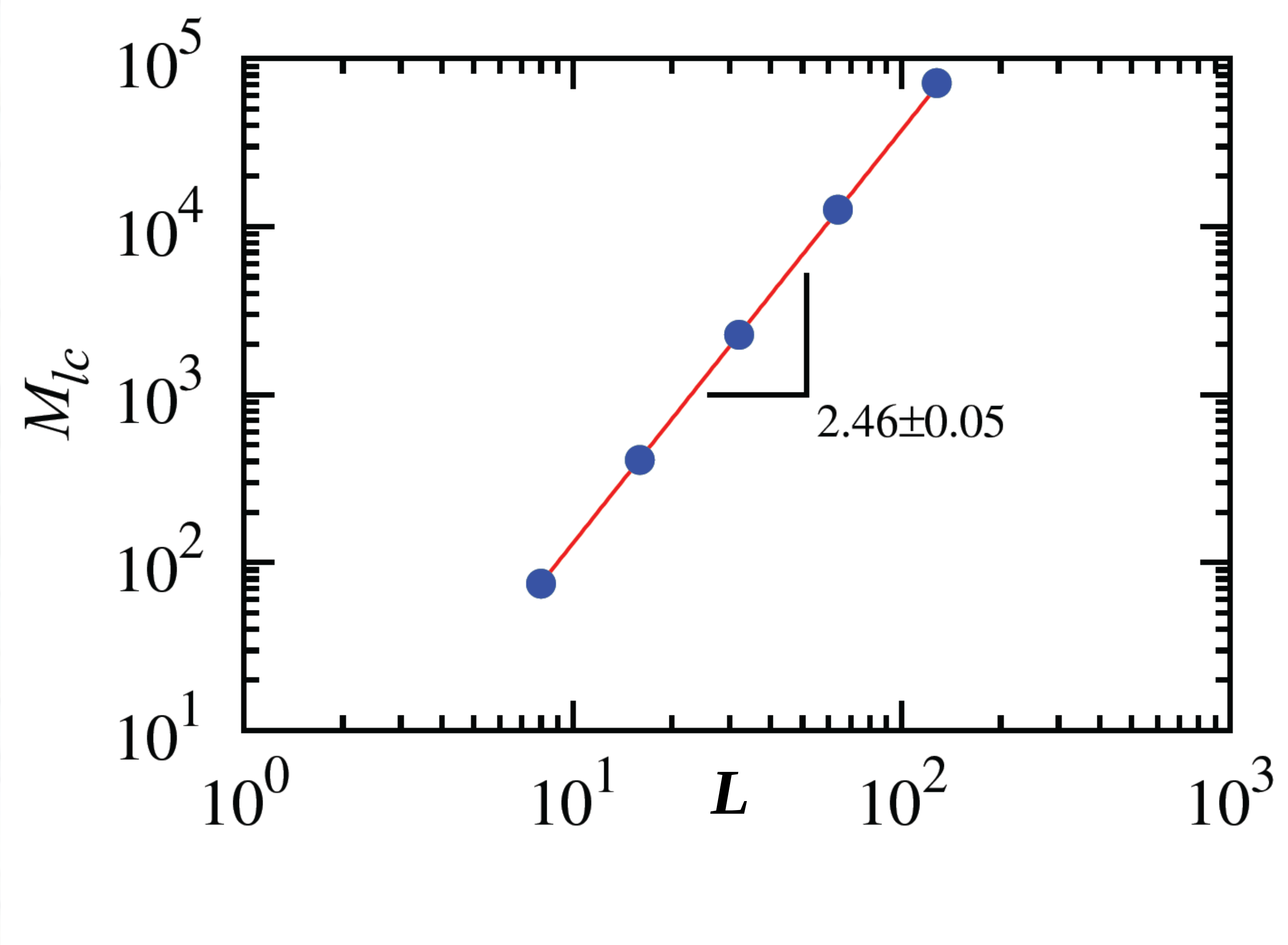
Julian Schrenk

K.J. Schrenk, N.A.M. Araújo, and H.J.H., Phys. Rev. E, 84, 041136 (2011)

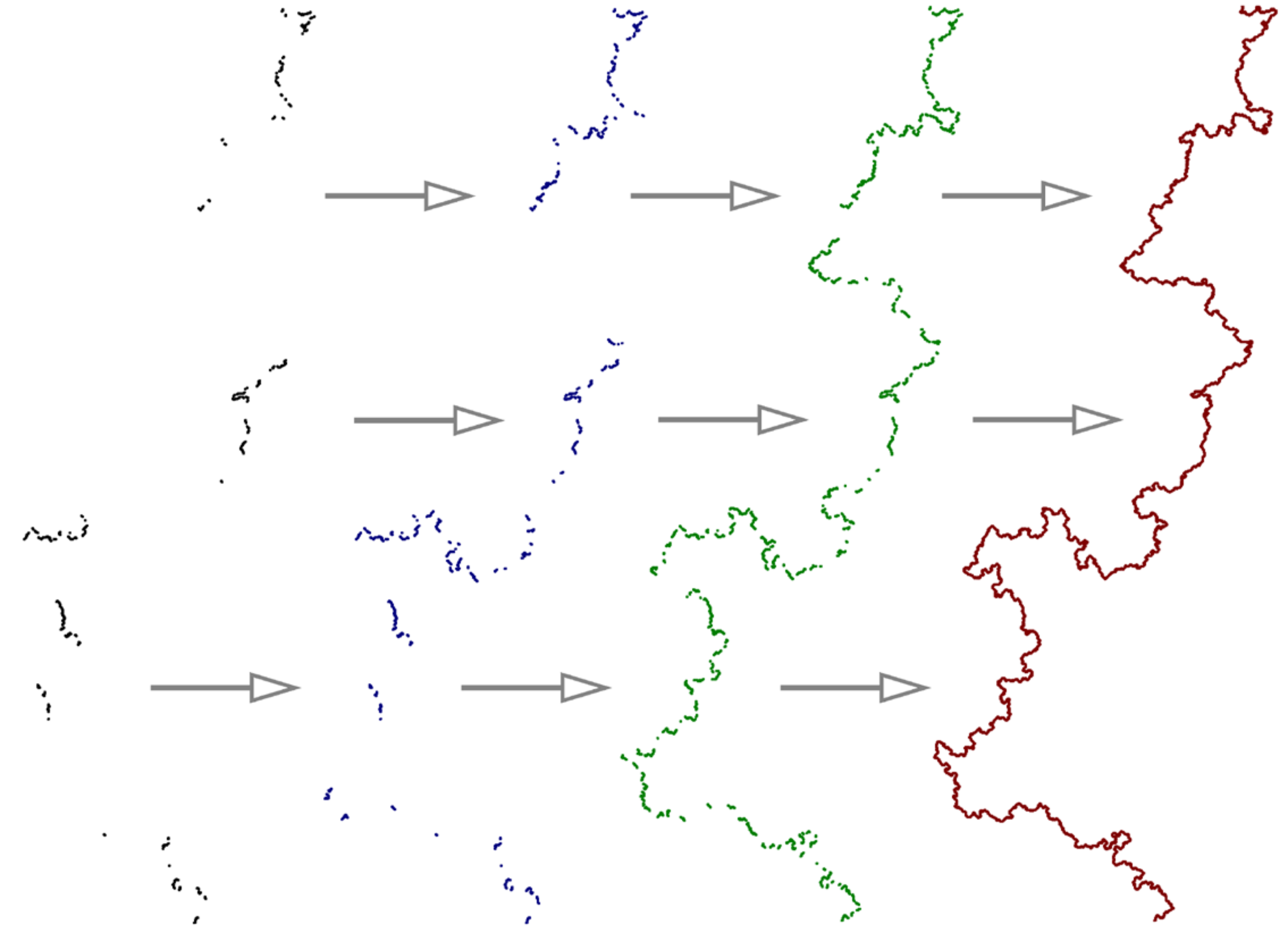




Largest cluster model in 3D

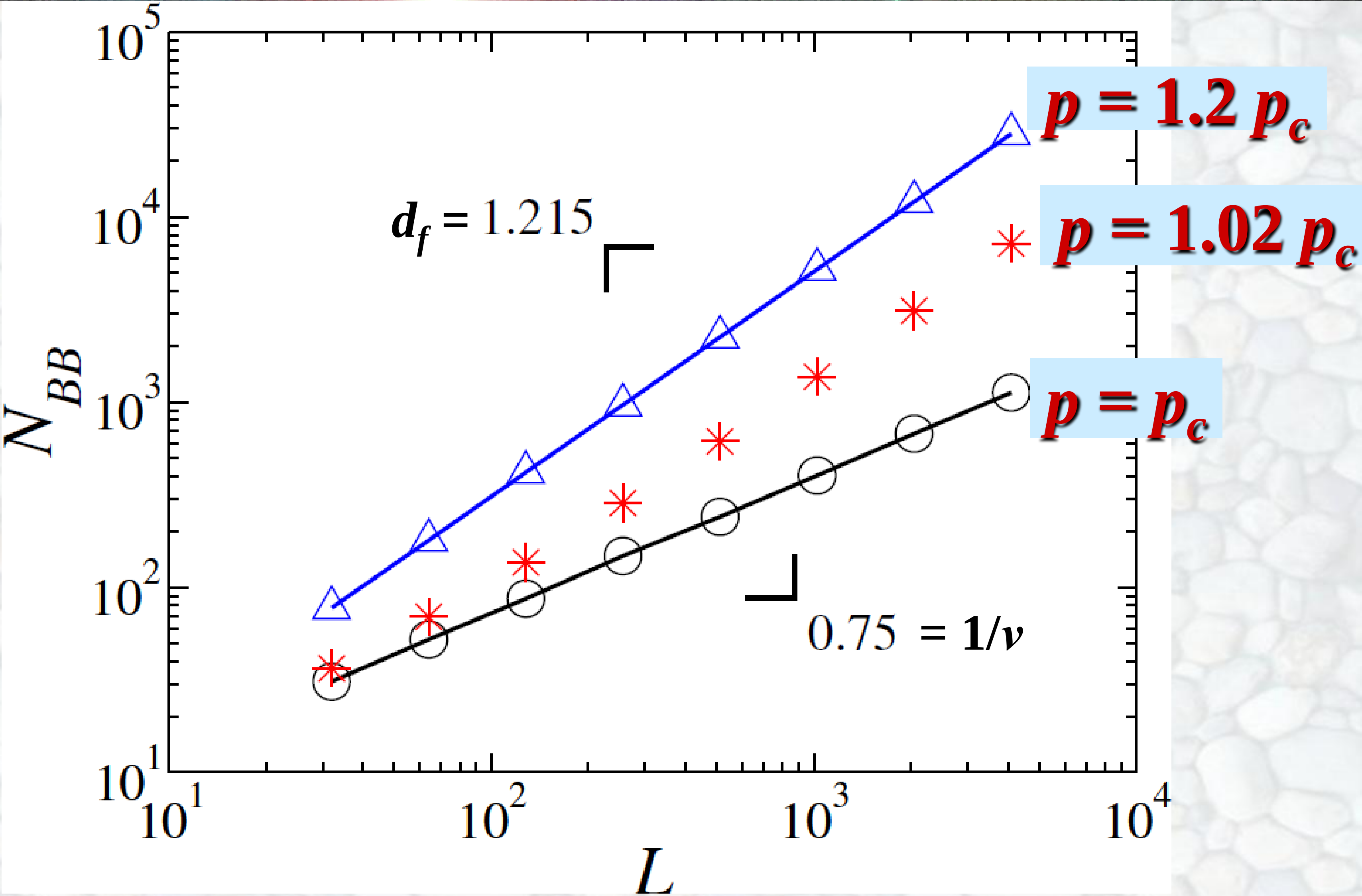


Bridge Percolation



$p = p_c$ | $p = 1.01\, p_c$ | $p = 1.05\, p_c$ | $p = 1$

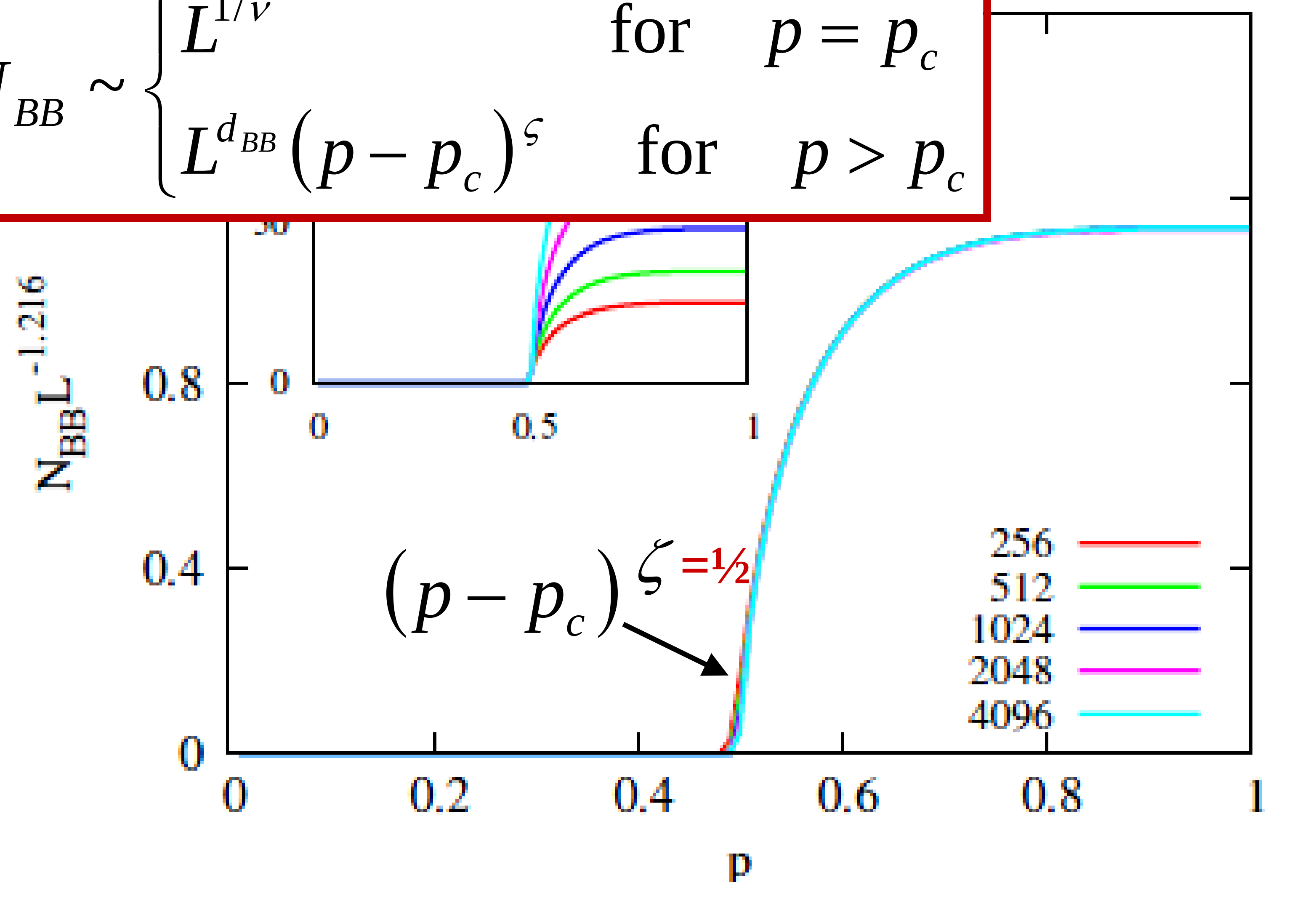
Bridge Percolation



Bridge Percolation

$$N_{BB} \sim \begin{cases} L^{1/\nu} & \text{for} \quad p = p_c \\ L^{d_{BB}} (p - p_c)^{\varsigma} & \text{for} \quad p > p_c \end{cases}$$

N_{BB} is number of bridge bonds.



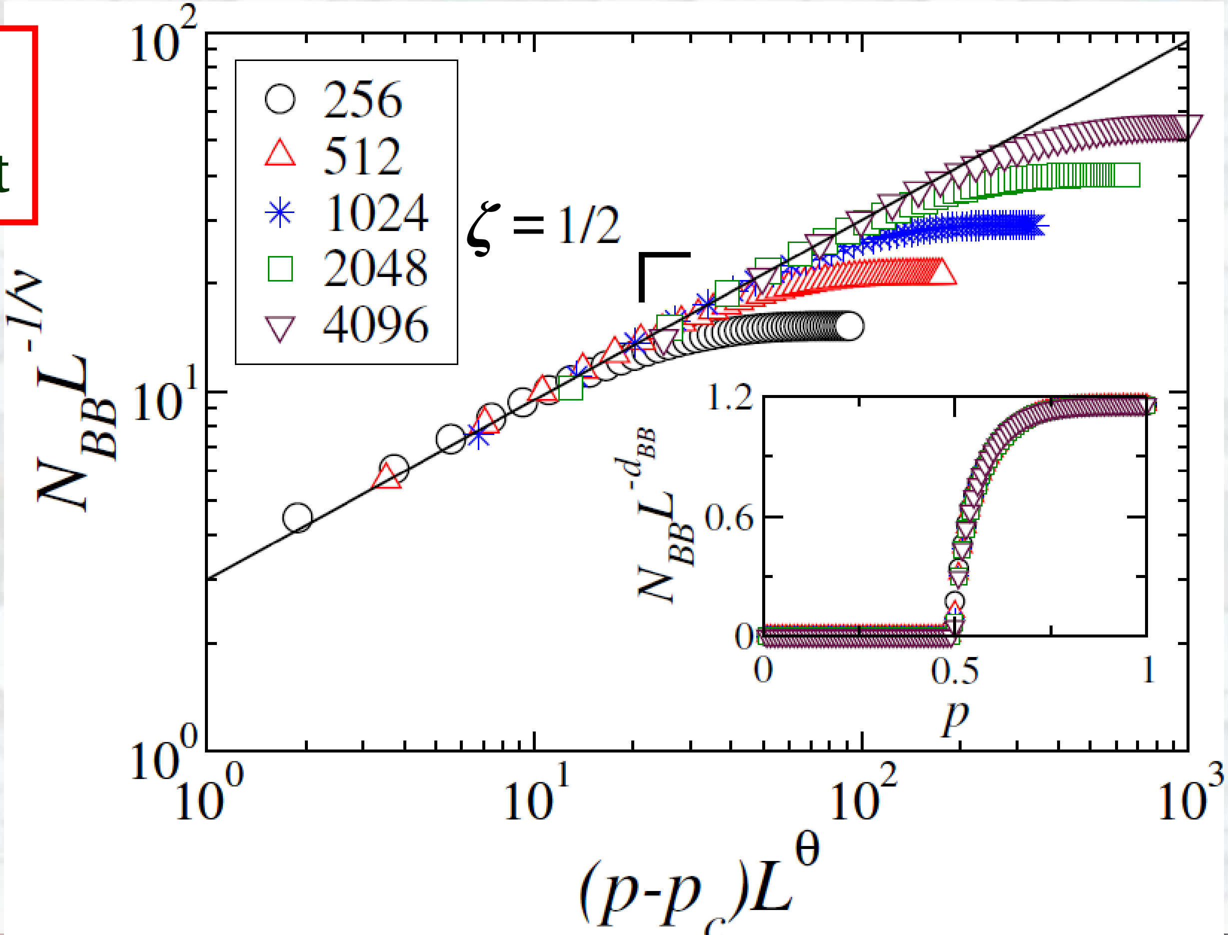
Bridge Percolation

$$N_{BB}(p,L) = L^{\frac{1}{\nu}}\mathcal{F}\left[(p-p_c)L^{\theta}\right] = L^{d_{BB}}\tilde{\mathcal{F}}\left[(p-p_c)L^{\varsigma}\right]$$

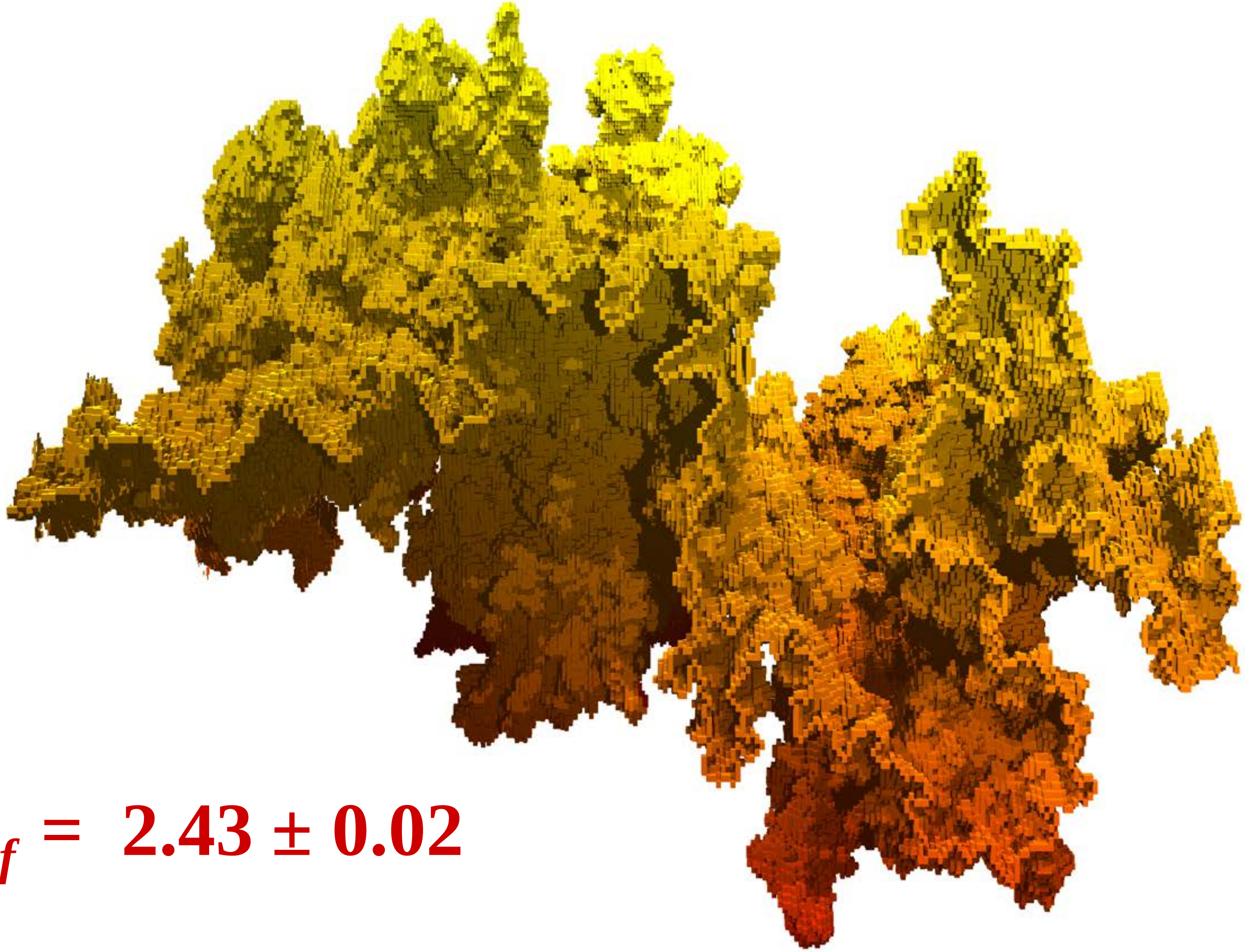
Tricritical scaling analogous to theta point

$$\theta = \zeta^{-1}\left(d_f - \frac{1}{\nu}\right)$$

$\theta = 0.94$

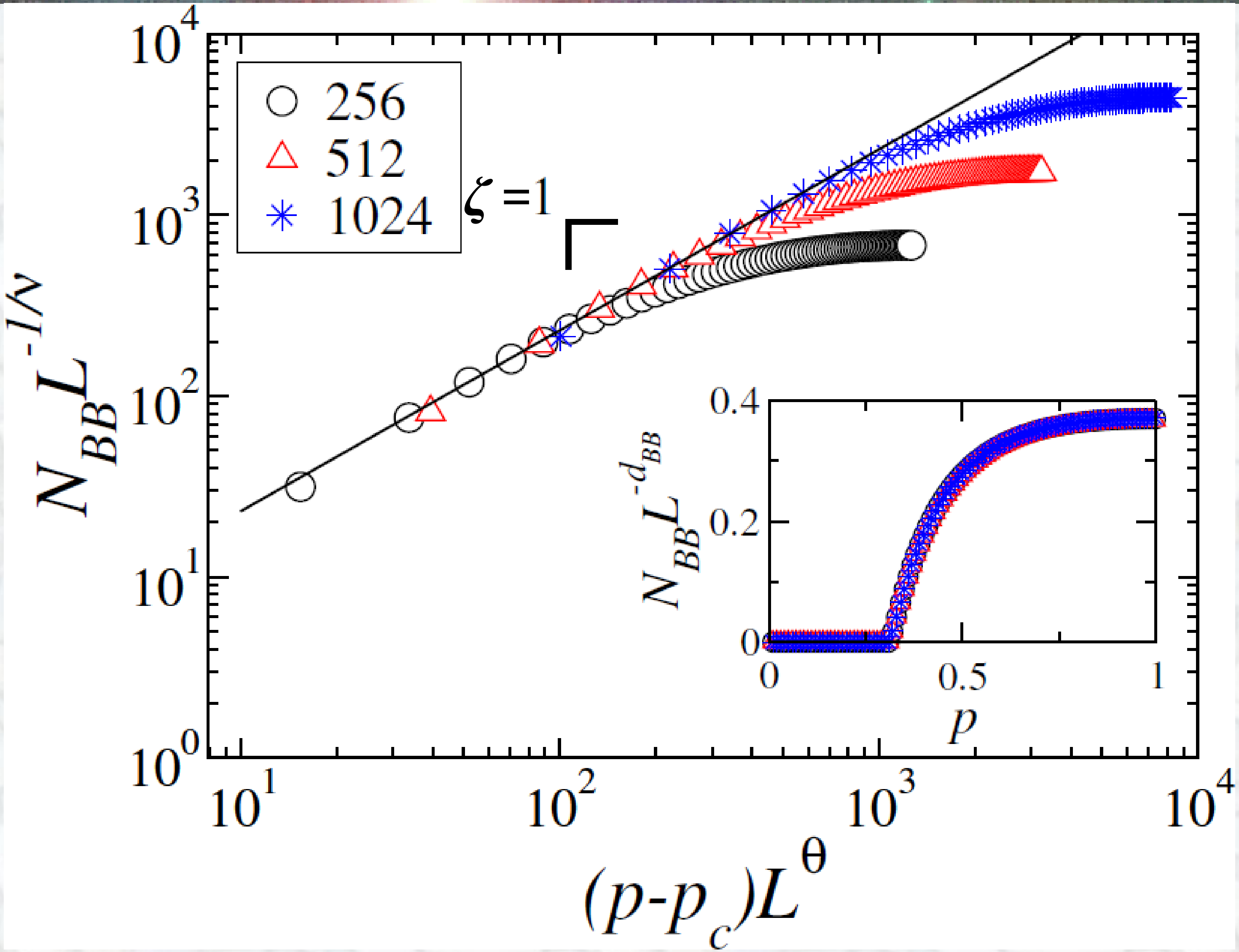


Bridge Percolation in 3D



$d_f = 2.43 \pm 0.02$

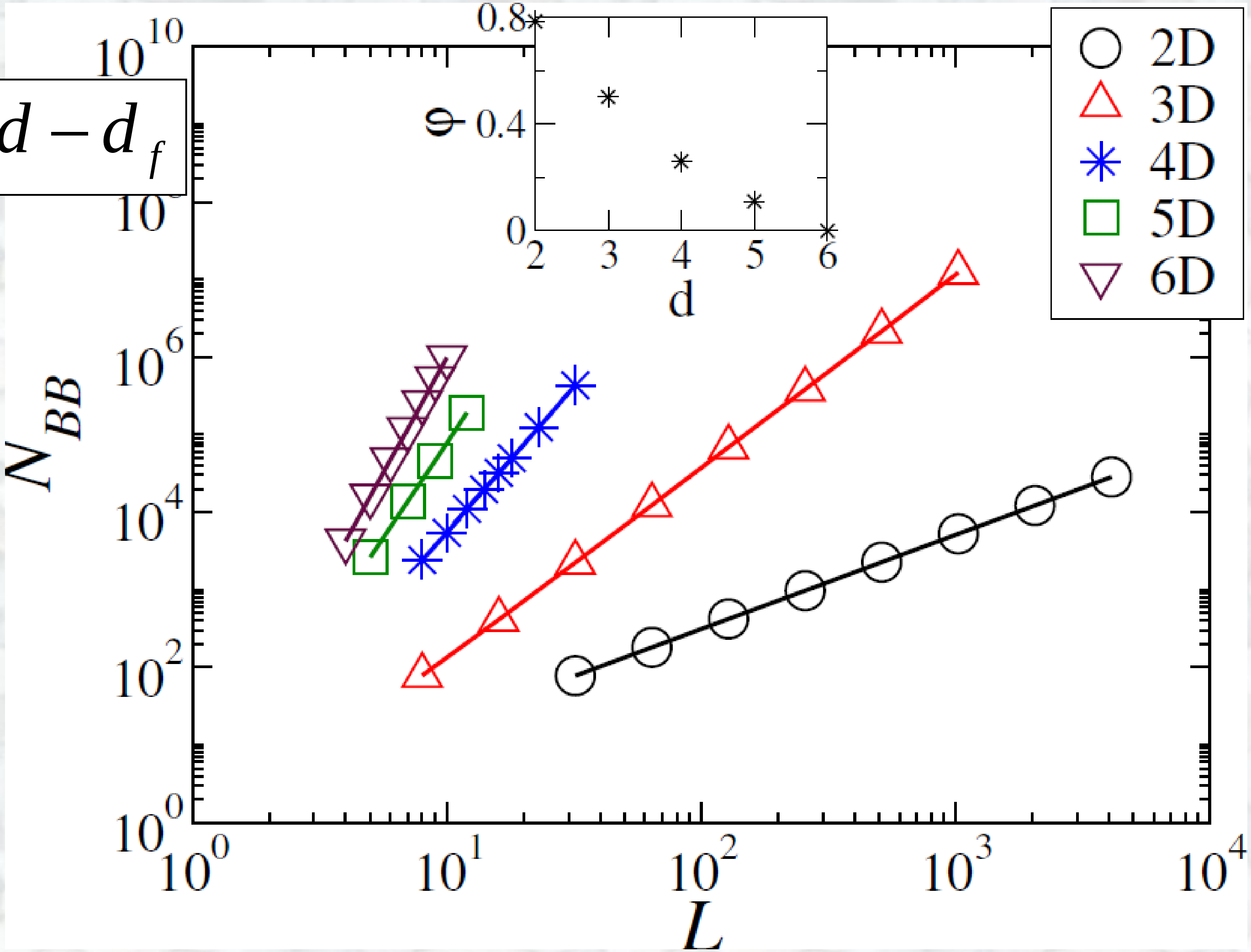
Bridge Percolation in 3D



$\theta = 1.36$

Bridge Percolation $d = 2 - 6$

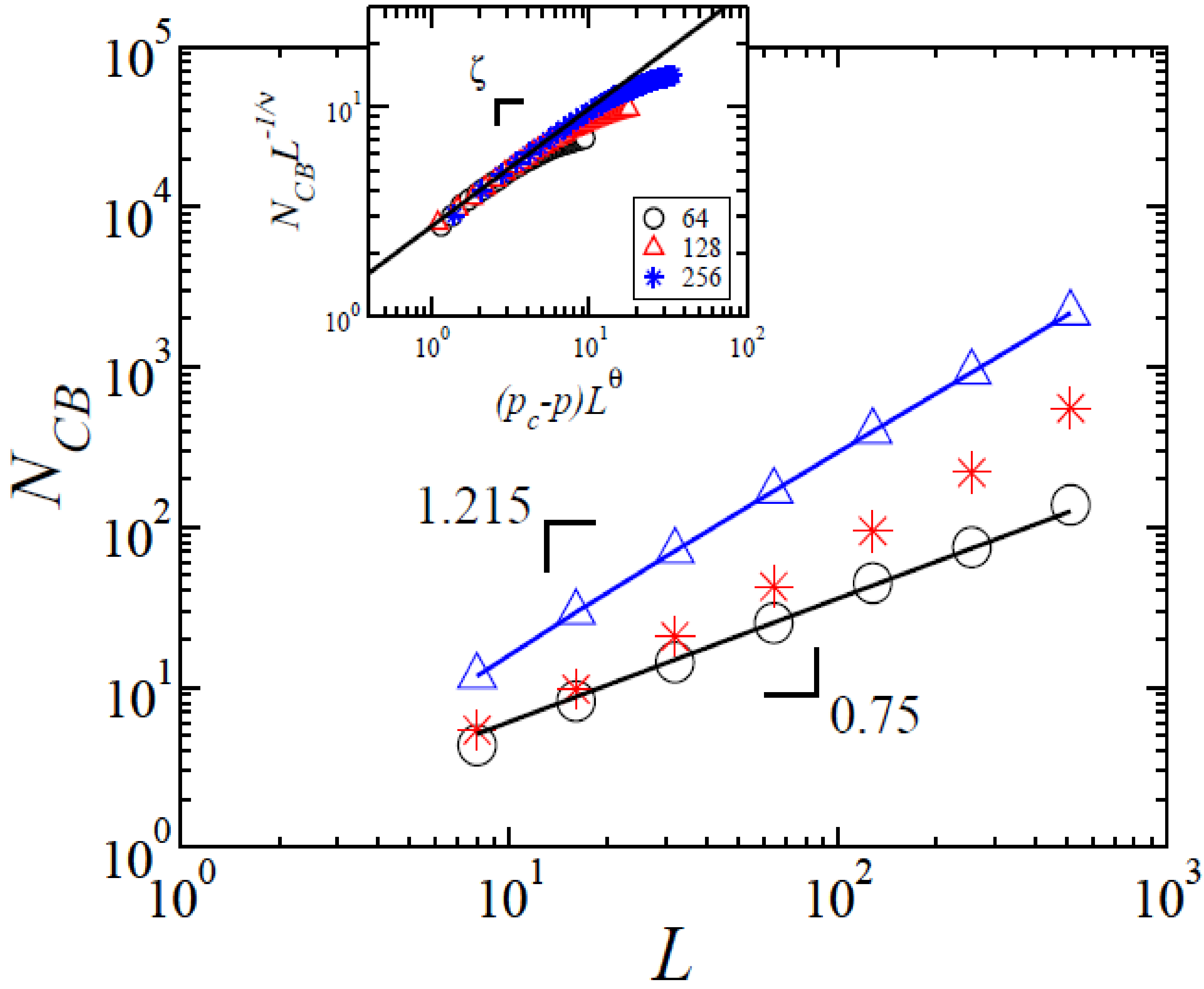
$$\varphi = d - d_f$$



Above the upper critical dimension $d_c = 6$ the set of bridges is dense.

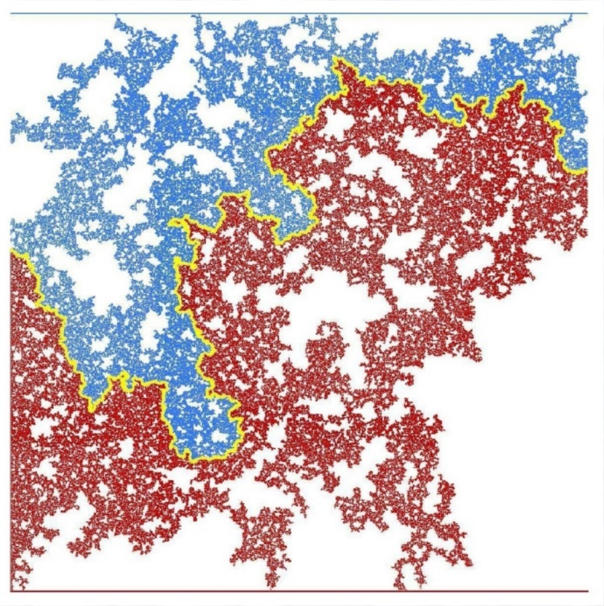
Cutting bonds

If one starts from a fully occupied lattice and removes bonds except if they are cutting bonds
in 2d they have the same behavior as the bridges before (same exponents).
In higher dimension the exponents are different.



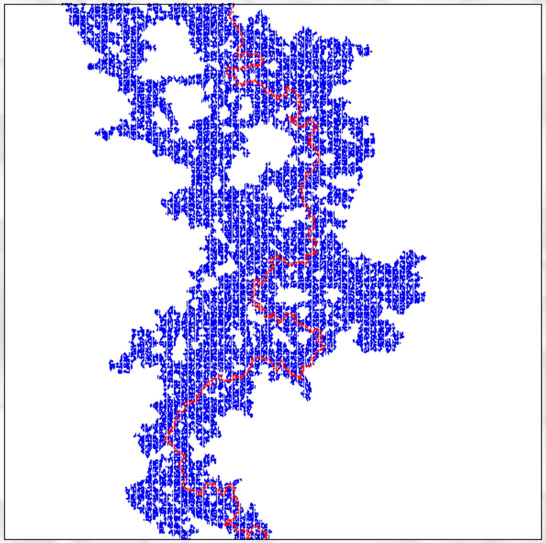
Same fractal dimension

watersheds

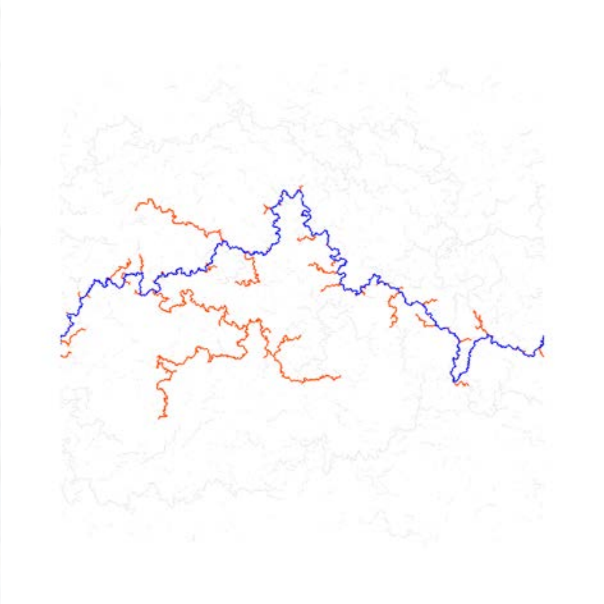


E. Fehr, J.S. Andrade Jr., S.D. da Cunha, L.R. da Silva, H.J.H., D. Kadau, C.F. Moukarzel, E.A. Oliveira, J. Stat. Mech. P09007 (2009)

shortest path on loop-less percolation



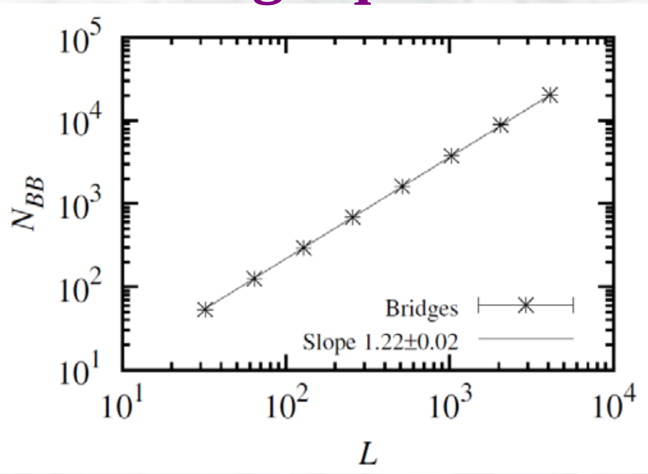
optimal path crack



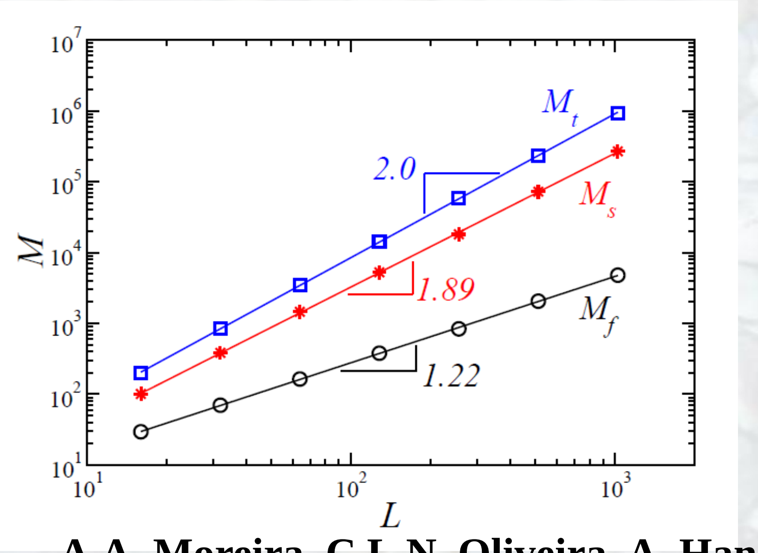
J.S. Andrade Jr., E. Oliveira, A. Moreira and HJH, Phys.Rev.Lett. 103, 225503 (2009)

Same fractal dimension

Two invading liquids touching

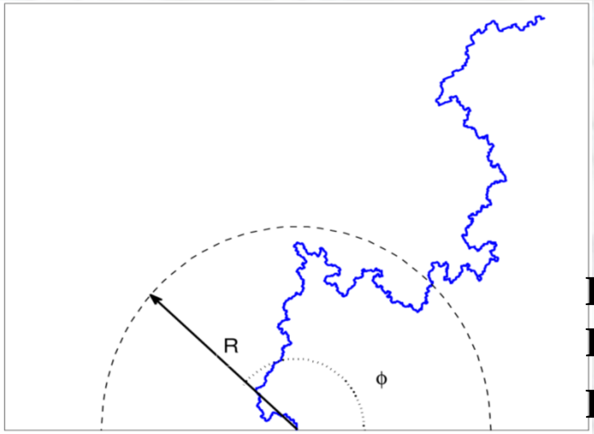


Fuses in infinite disorder



A.A. Moreira, C.L.N. Oliveira, A. Hansen, N.A.M. Araújo, H.J.H., J.S. Andrade Jr, Phys. Rev. Lett. 109, 255701 (2012)

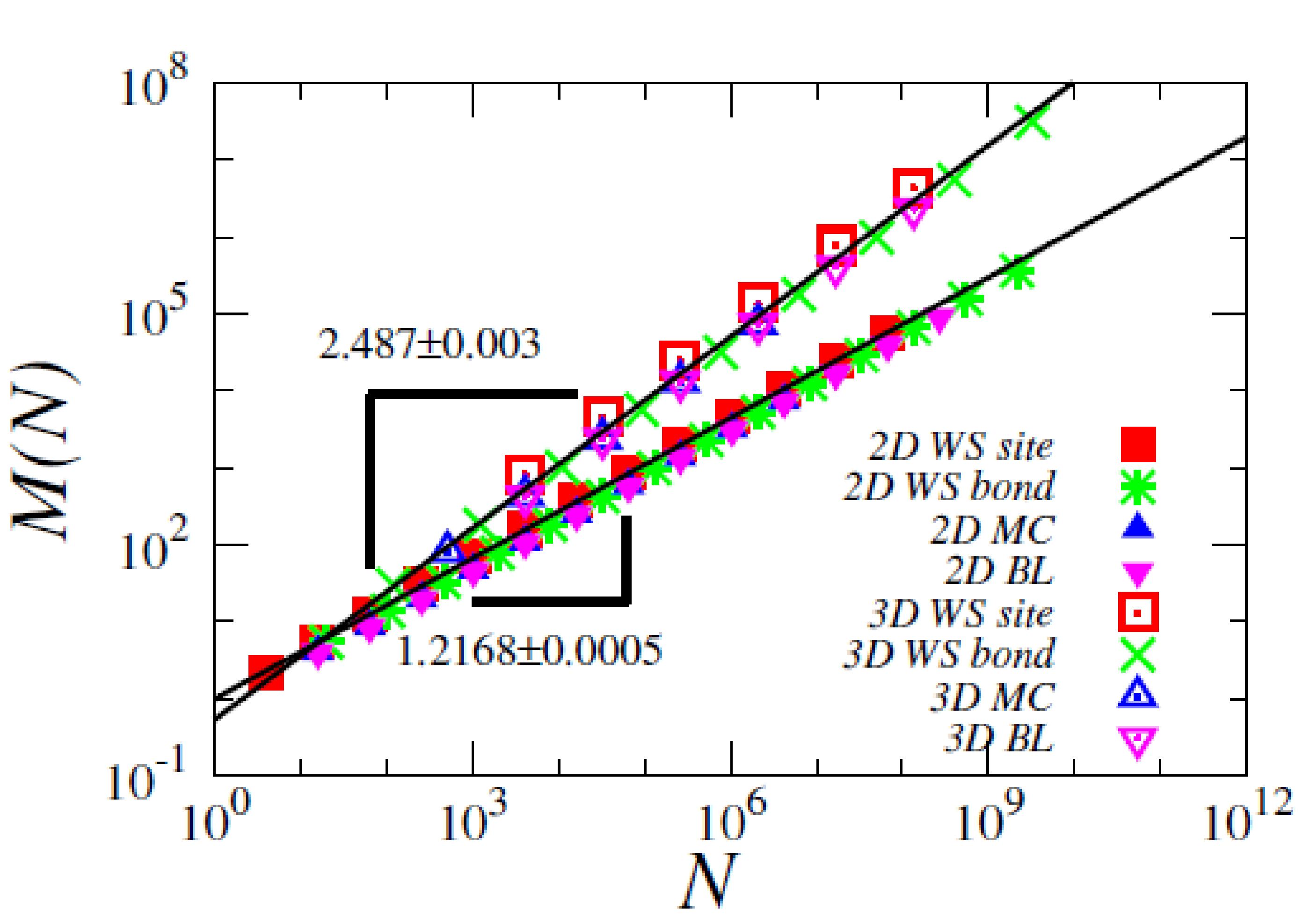
Schramm-Loewner Evolution (SLE)



E. Daryaei, N. A. M. Araújo, K. J. Schrenk, S. Rouhani and H. J. H.

Phys. Rev. Lett. 109, 218701 (2012)

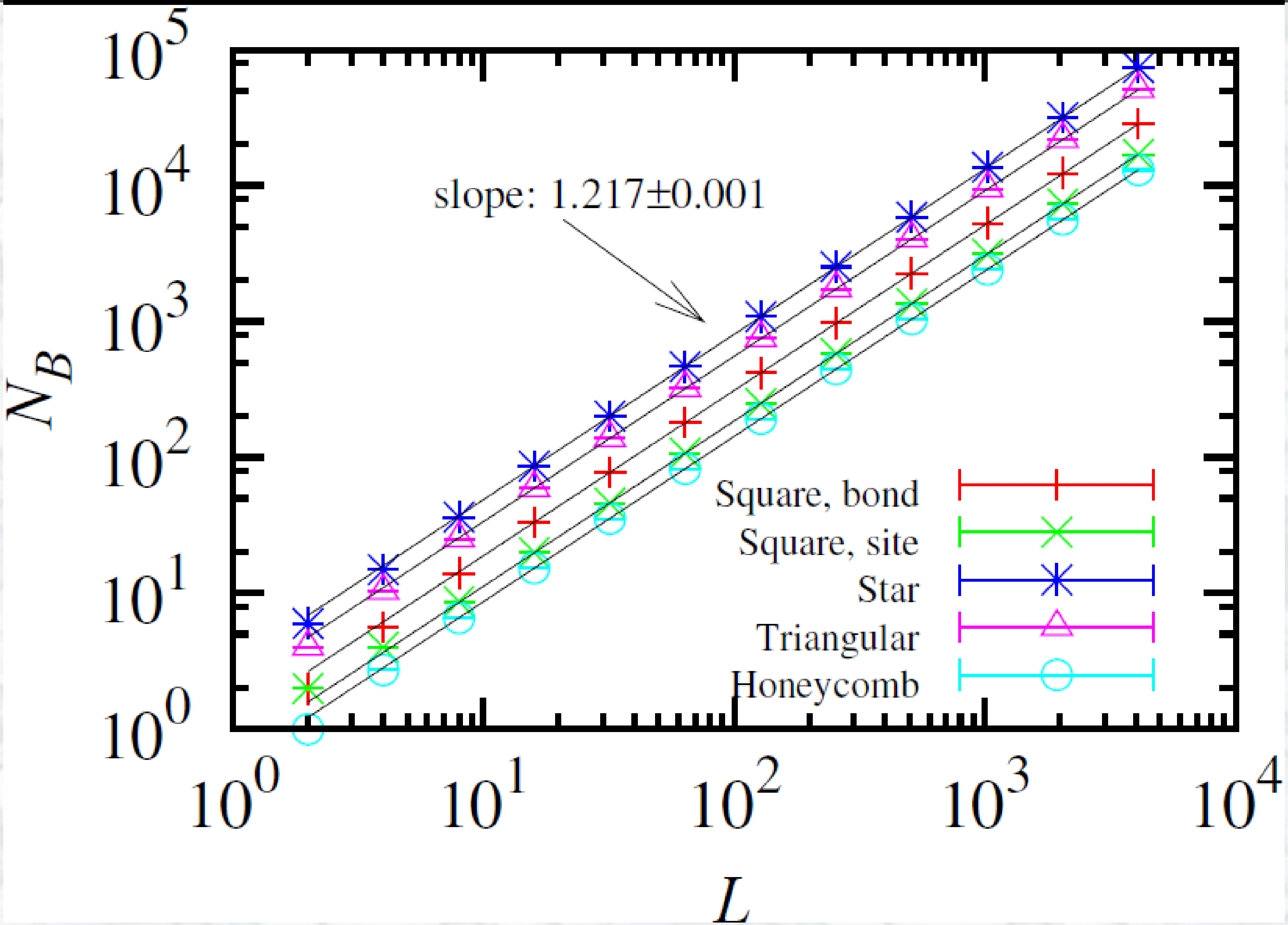
High precision calculation



E. Fehr, K.J. Schrenk, N.A.M. Araújo, D. Kadau, P. Grassberger, J.S. Andrade Jr., H.J.H.

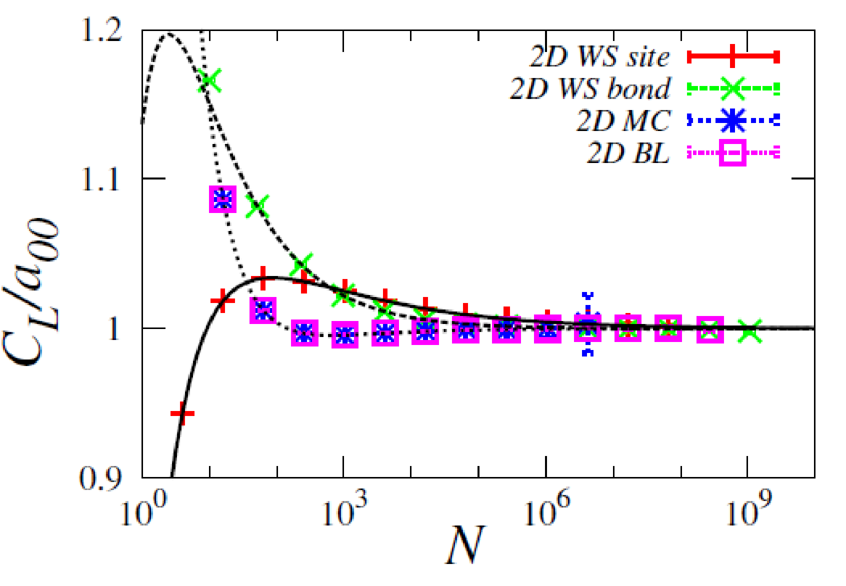
Phys. Rev.E 86, 011117(2012)

Universality

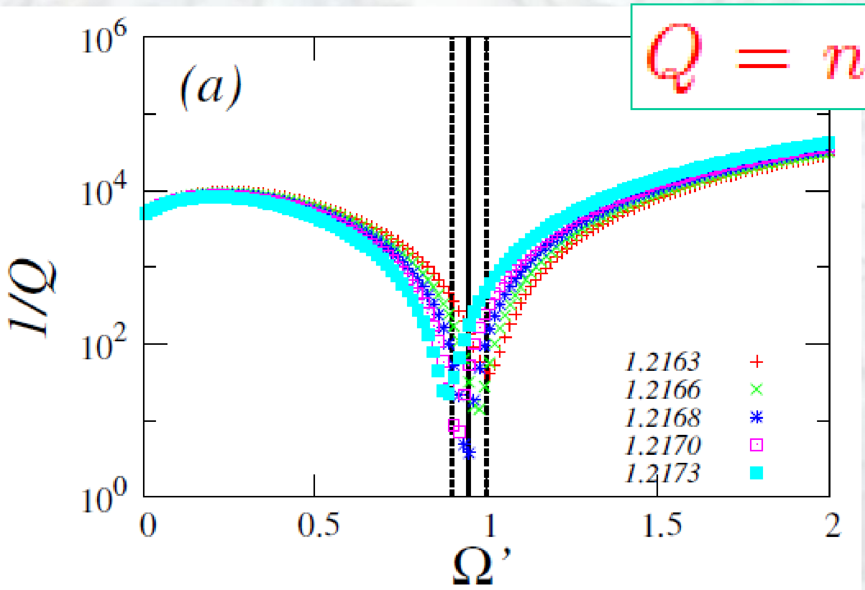


Corrections to scaling

$$C_L^{2D} = a_{00} + a_{11}L^{-\omega} + a_{21}L^{-\Omega} + a_{22}L^{-\Omega-1}$$



$$Q = n/\chi^2$$



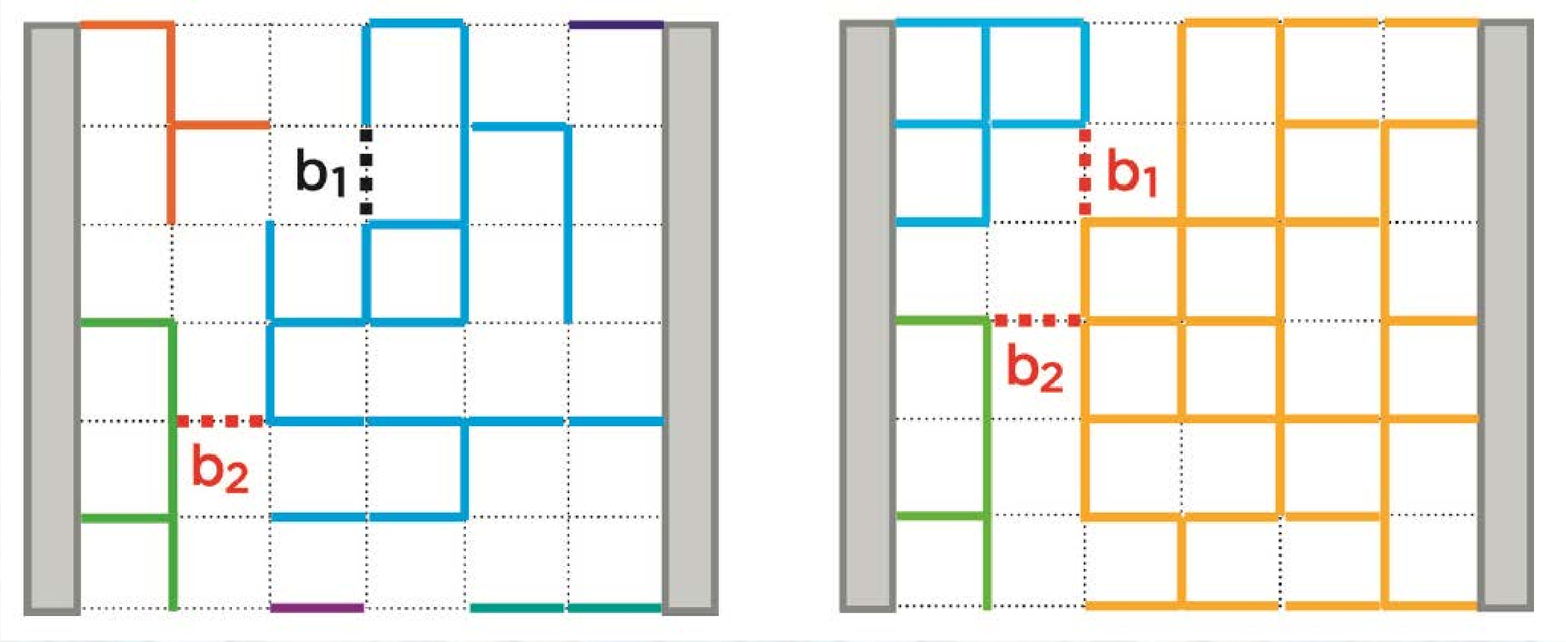
model	d	d_f	Ω
WS bond	2	1.2168±0.0005	0.95±0.05
WS site	2	1.21705±0.00075	0.91±0.19
BL	2	1.21655±0.0015	0.87±0.08
MC	2	1.21655±0.0045	0.86±0.11
WS bond	3	2.4865±0.0025	0.96±0.10
WS site	3	2.4865±0.0025	0.98±0.09
BL	3	2.4878±0.0025	1.06±0.16

Spanning cluster avoiding model

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

Choose *m* unoccupied bonds and occupy randomly one which is not a bridge, if all are bridges then choose randomly one of these bridges.

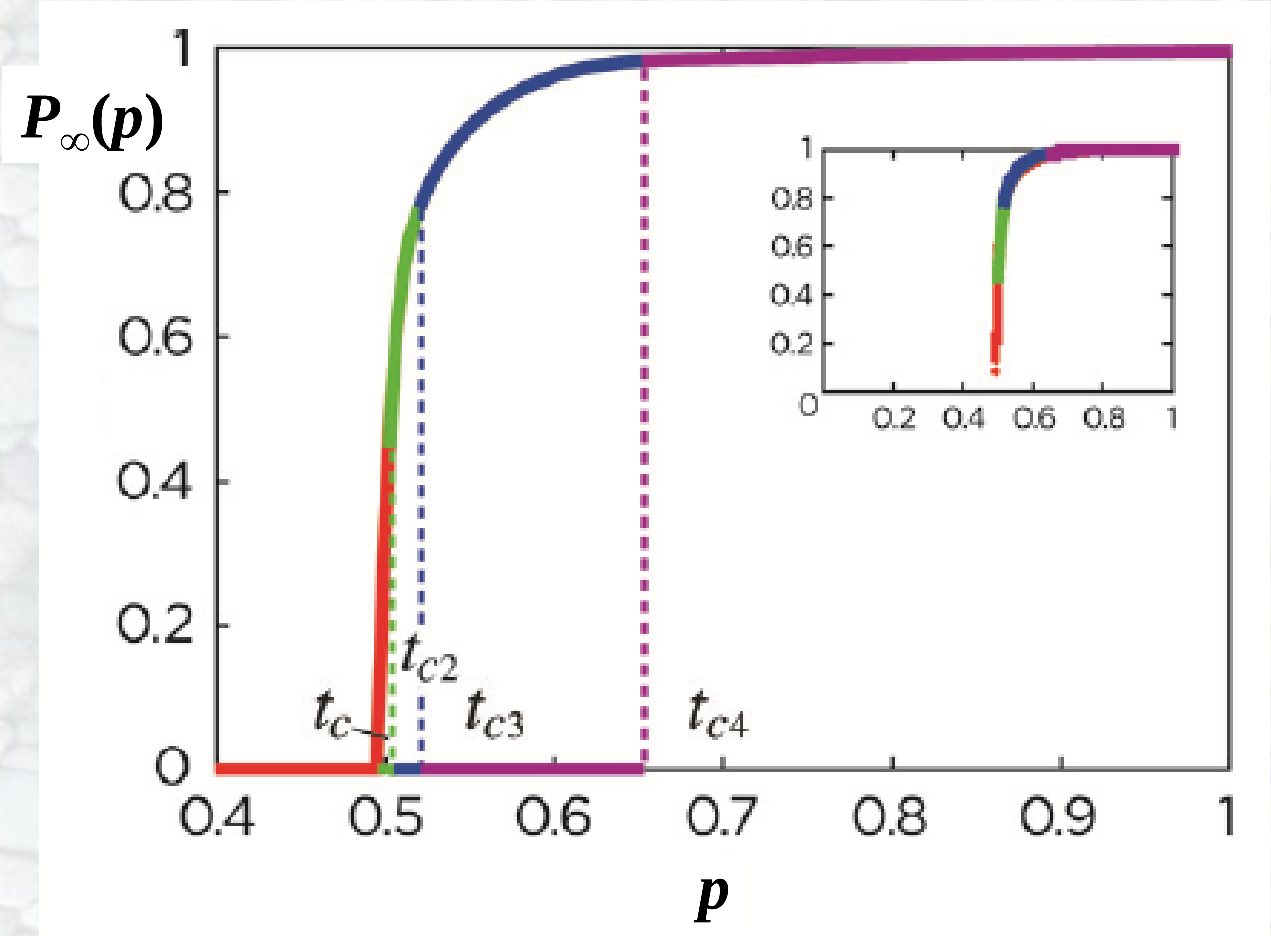
***m* = 2**



Spanning cluster avoiding model

For finite systems there is a jump for *m* > 1.

fraction of bonds in the spanning cluster



L = 1000

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

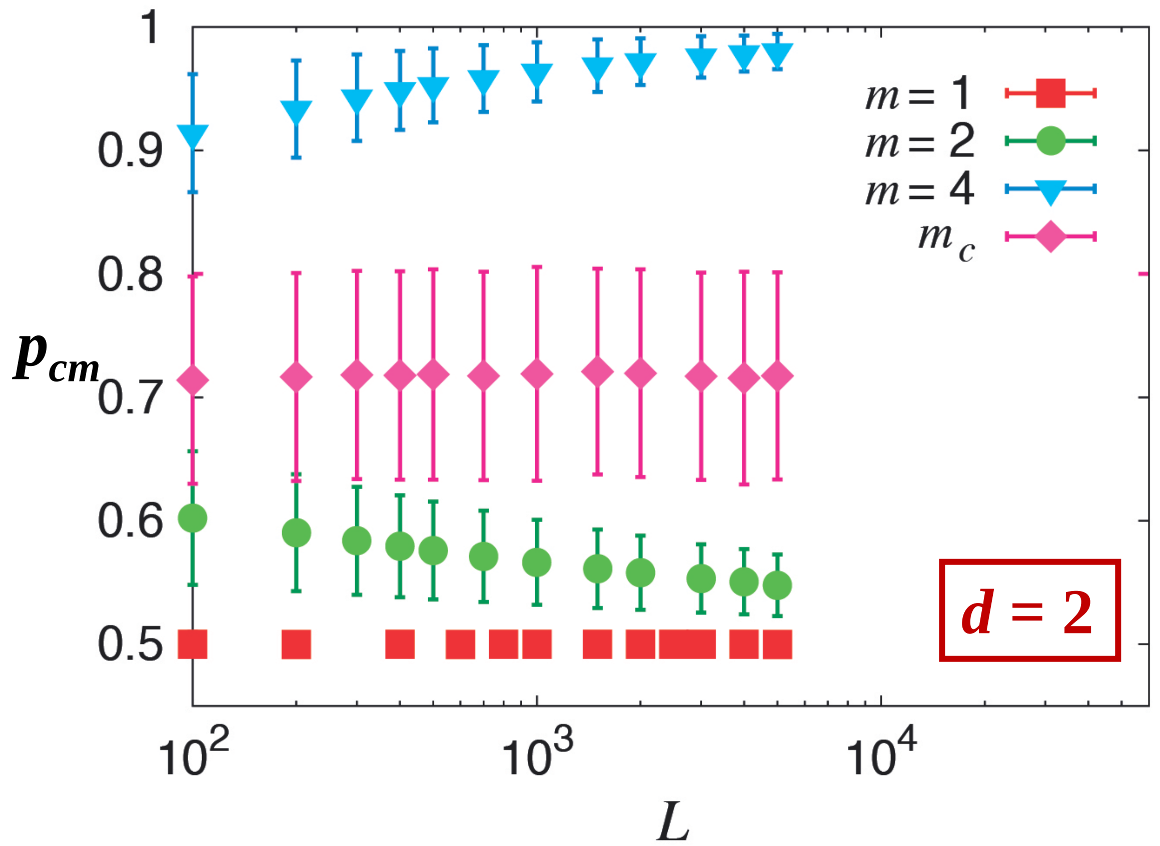
Spanning cluster avoiding model

At each dimension d there exists an m_c so that for increasing system size L the transition goes to

$p_c = 0.5$ for $m < m_c$

and to

$p_c = 1$ for $m > m_c$.



$$m_c(2) \approx 2.55 \pm 0.01 \qquad m_c(3) = 5.98 \pm 0.07 \qquad m_c(4) = 16.99 \pm 5.23$$

Y. S. Cho, S. Hwang, H.J.H., and B. Kahng, Science, 339, 1185 (2013)

Spanning cluster avoiding model

$N_b = d\, L^d$ is the number of bonds

$$N_{BB} \sim \begin{cases} L^{1/\nu} & \text{for} \quad p = p_c \\ L^{d_{BB}} (p - p_c)^{\varsigma} & \text{for} \quad p > p_c \end{cases}$$

probability to have *m* bridge bonds:

$$q(p,m) = \left[\frac{N_{BB}}{N_b (1-p)} \right]^m \sim N_b^{-m/m_c} \left[\frac{(p-p_c)^{\varsigma}}{1-p} \right]^m$$

$$\Rightarrow \quad m_c(d) = \frac{d}{d - d_{BB}} \quad \Rightarrow$$

For *d* > 6 the transition is always continuous.

Spanning cluster avoiding model

One can also show analytically that:

for $m < m_c$

$$p_{cm}(N) - p_c \sim N^{-1/\bar{\nu}_<}$$

$$1/\bar{\nu}_< = (1 - m/m_c)/(m\zeta + 1),$$

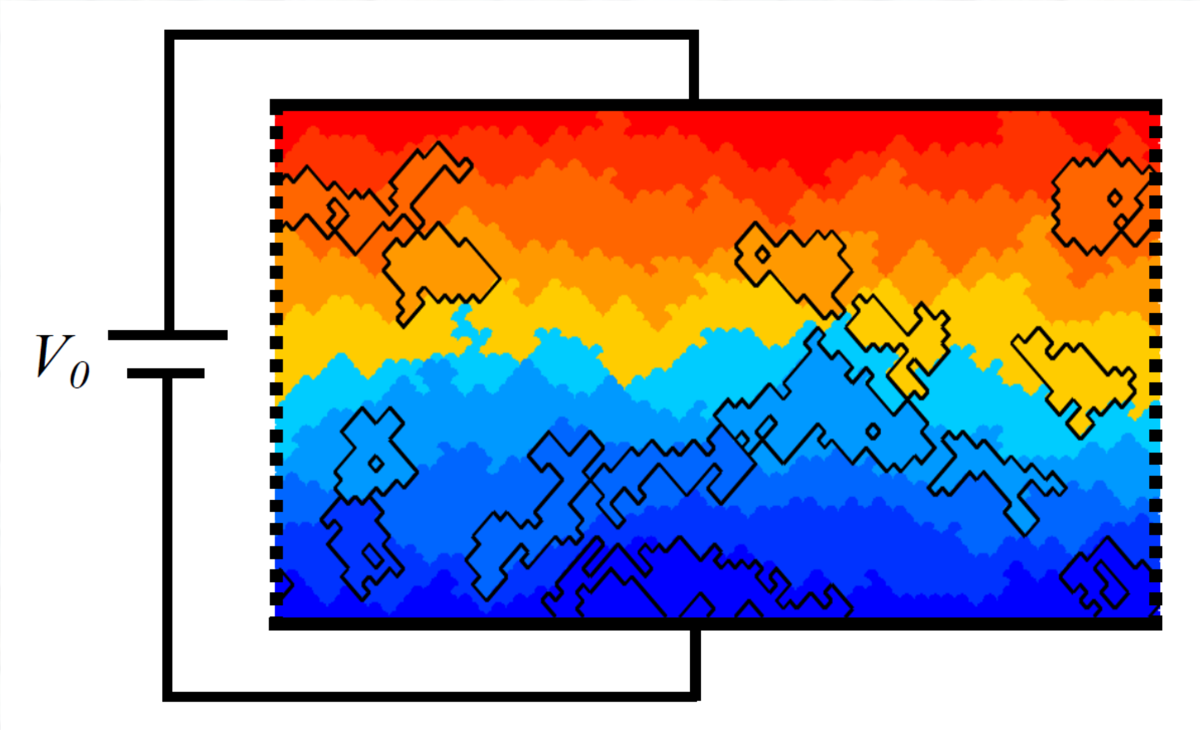
for $m > m_c$

$$1 - p_{cm}(N) \sim N^{-1/\bar{\nu}_>}$$

$$1/\bar{\nu}_> = (m/m_c - 1)/(m - 1)$$

Metallic Breakdown

Deposition of metallic particles on a dielectric surface



$q = 10$
$L = 128$
$\gamma = 0.1$
$p = 0.57$

C.L.N.Oliveira, N.A.M. Araújo, J.S. Andrade Jr., H.J.H.

Phys.Rev. Lett. 113, 155701 (2014)

Metallic Breakdown

Metallic particles can be adsorbed on the surface and desorbed again.

Adsorption is weaker, the stronger the local field.

Probability to replace a resistance by a metallic bond is:

$$W = \frac{p}{q}\left[1-(1-q)\left(1-\left(\frac{\Delta V}{V_0}\right)^{\gamma}\right)\right]$$

$q > 1$ describes the relative deposition disadvantage due to the presence of the electric field.

For $\gamma = -\infty$ this is equivalent to classical bond percolation.

Metallic Breakdown

F. Gliozzi, Phys. Rev. E 66, 016115 (2002)

Simulate critical clusters of the *q*-state Potts model

(Kasteleyn-Fortuin or Coniglio-Klein or Swendsen-Wang clusters):

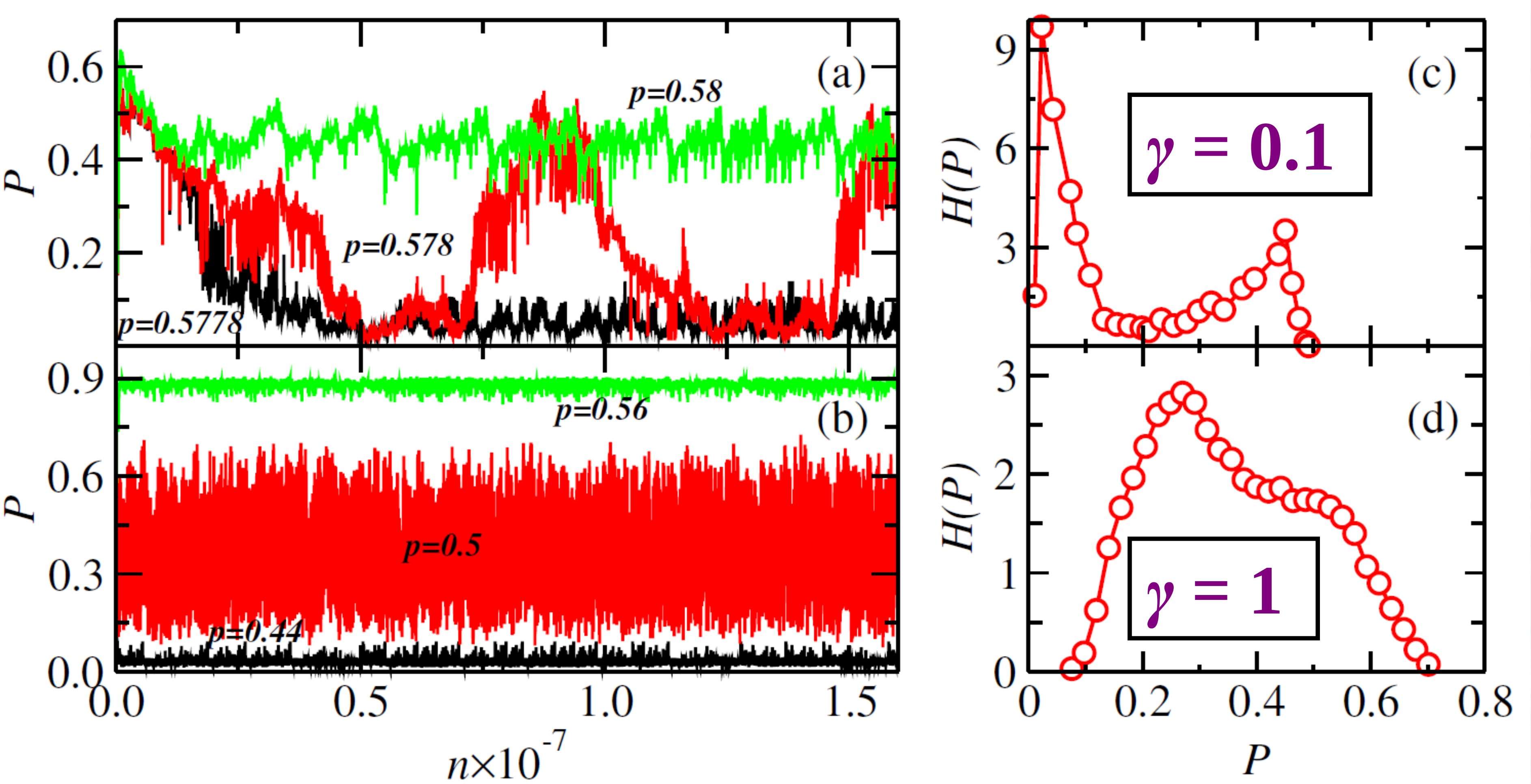
Be *x* a homogeneously distributed random number between 0 and 1.

1. **Occupy the bond, if $x < p/q$.**
2. **Make bond empty, if $x > p$.**
3. **Occupy if internal bond and make it empty, if it connects two metallic clusters, if $p/q < x < p$.**

When $\gamma = 0$ our model is identical to Gliozzi's method, because internal bonds are identified through $\Delta V = 0$.

Second order transition for $q \leq 4$ and first order transition for $q > 4$.

Metallic Breakdown



red is at transition

q = 10

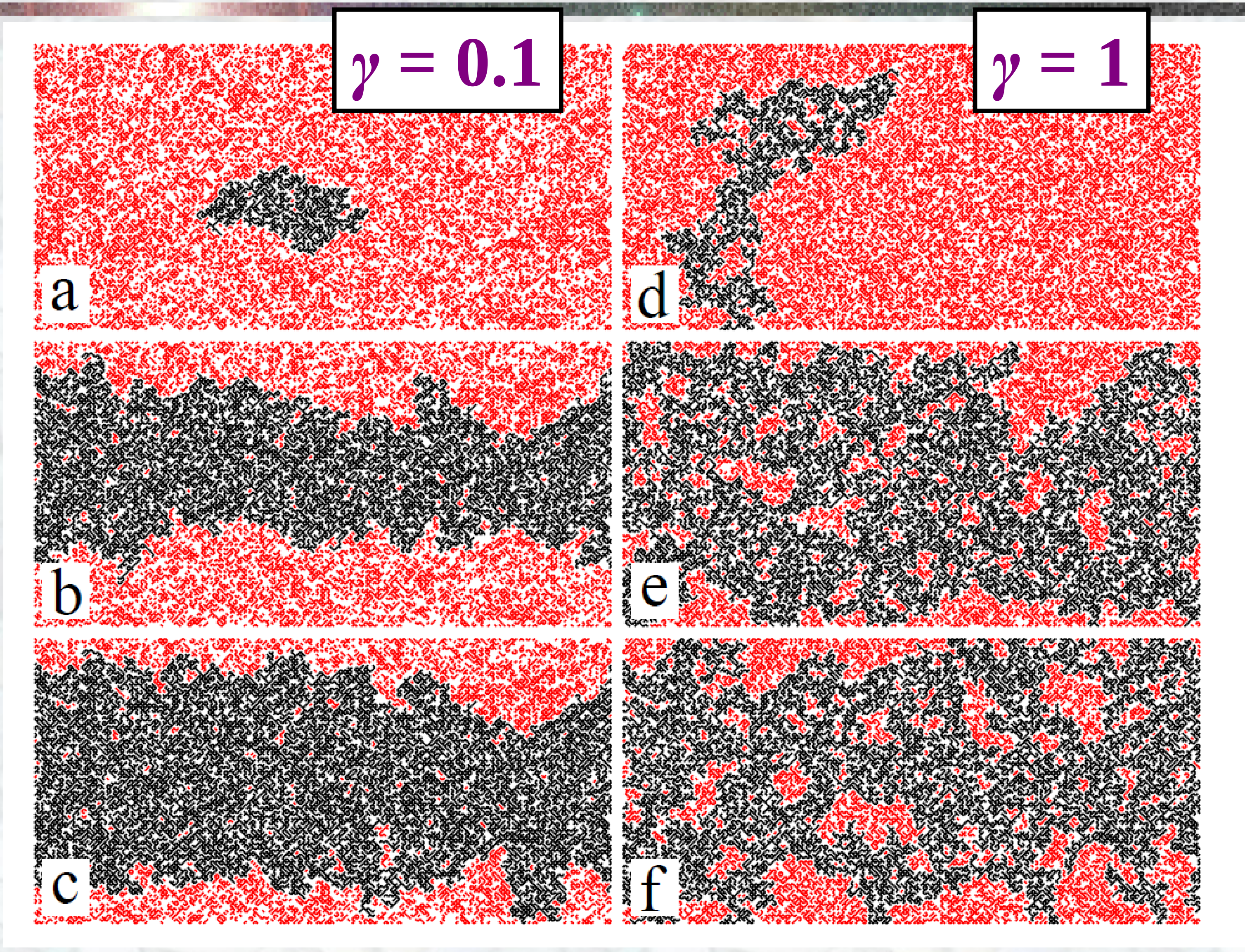
Largest Metallic Cluster

$q = 10$

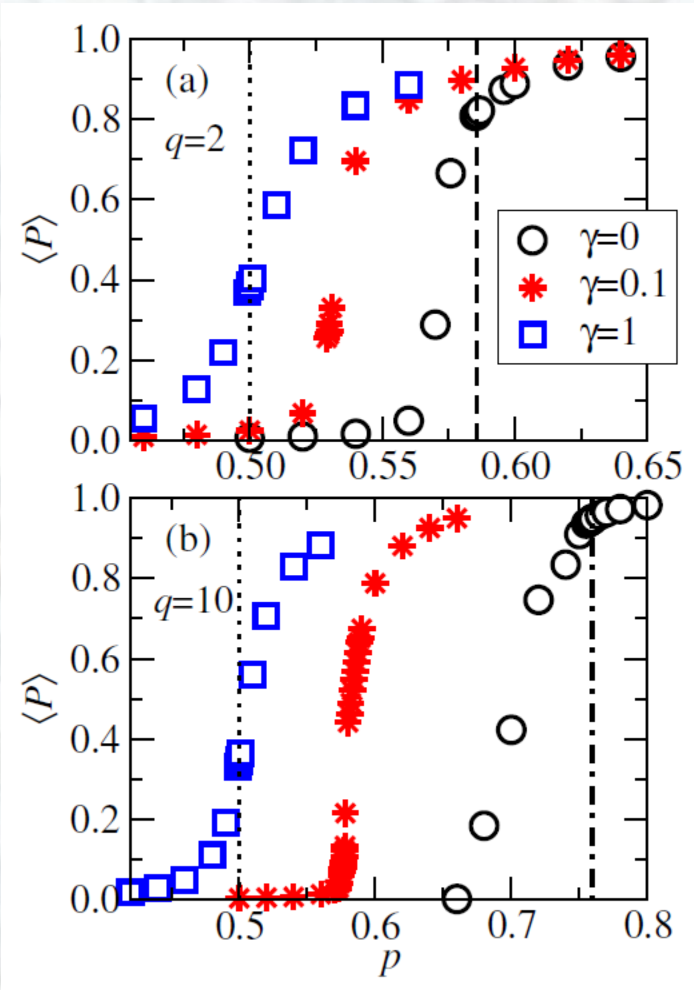
$p = 0.57$

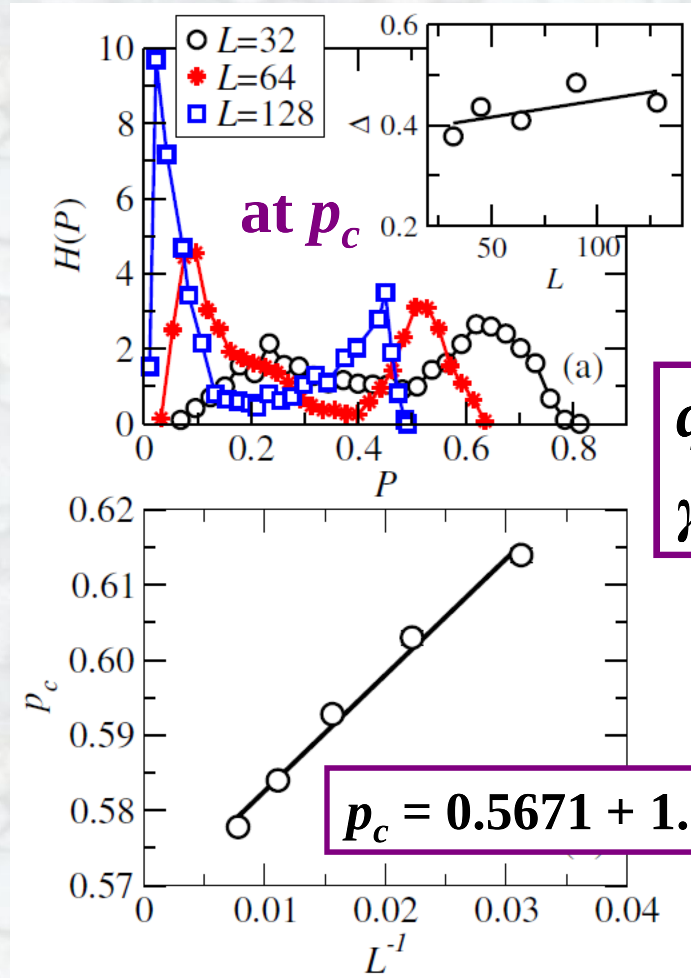
$p = 0.58$

$p = 0.59$



Metallic Breakdown





q = 10

γ = 0.1

$p_c = 0.5671 + 1.5463L^{-1}$

Connecting the Disconnected

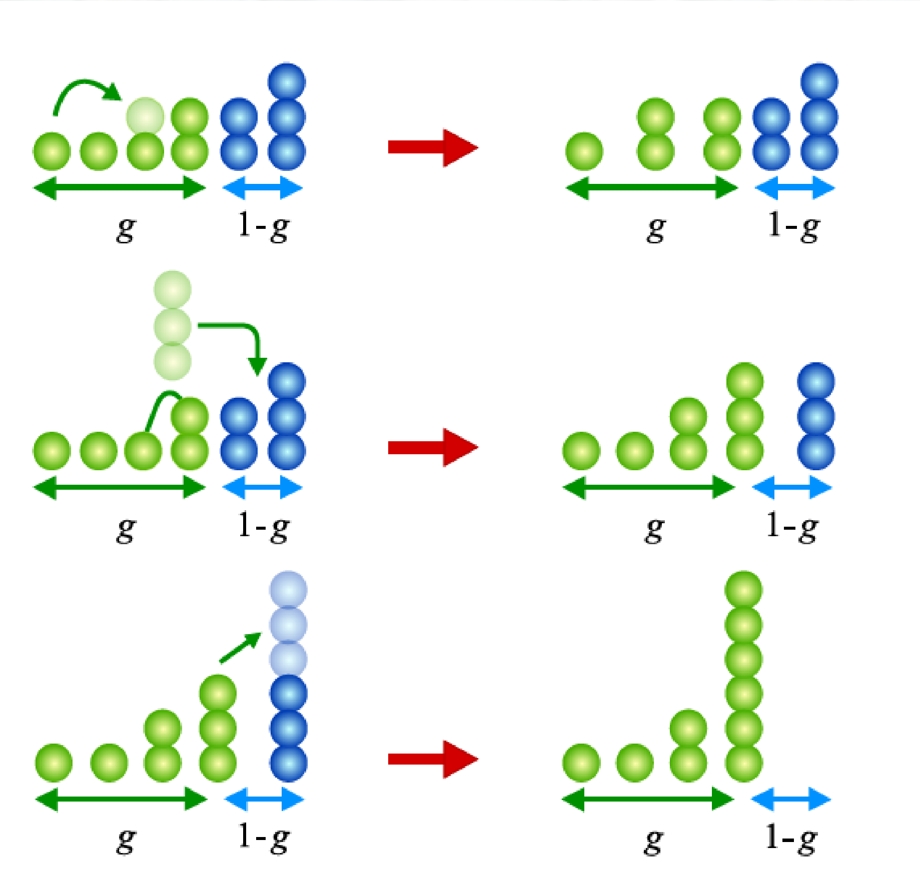
Young-Sul Cho

Byungnam Kahng

Y.S. Cho, J.S. Lee, H.J.H., B. Kahng, preprint 2015

Connecting the Disconnected

Connect randomly individuals but with a law imposing that every new connection must at least involve one individual belonging to the fraction *g* of the most disconnected population.



Y.S. Cho, J.S. Lee, H.J.H., B. Kahng, preprint 2015

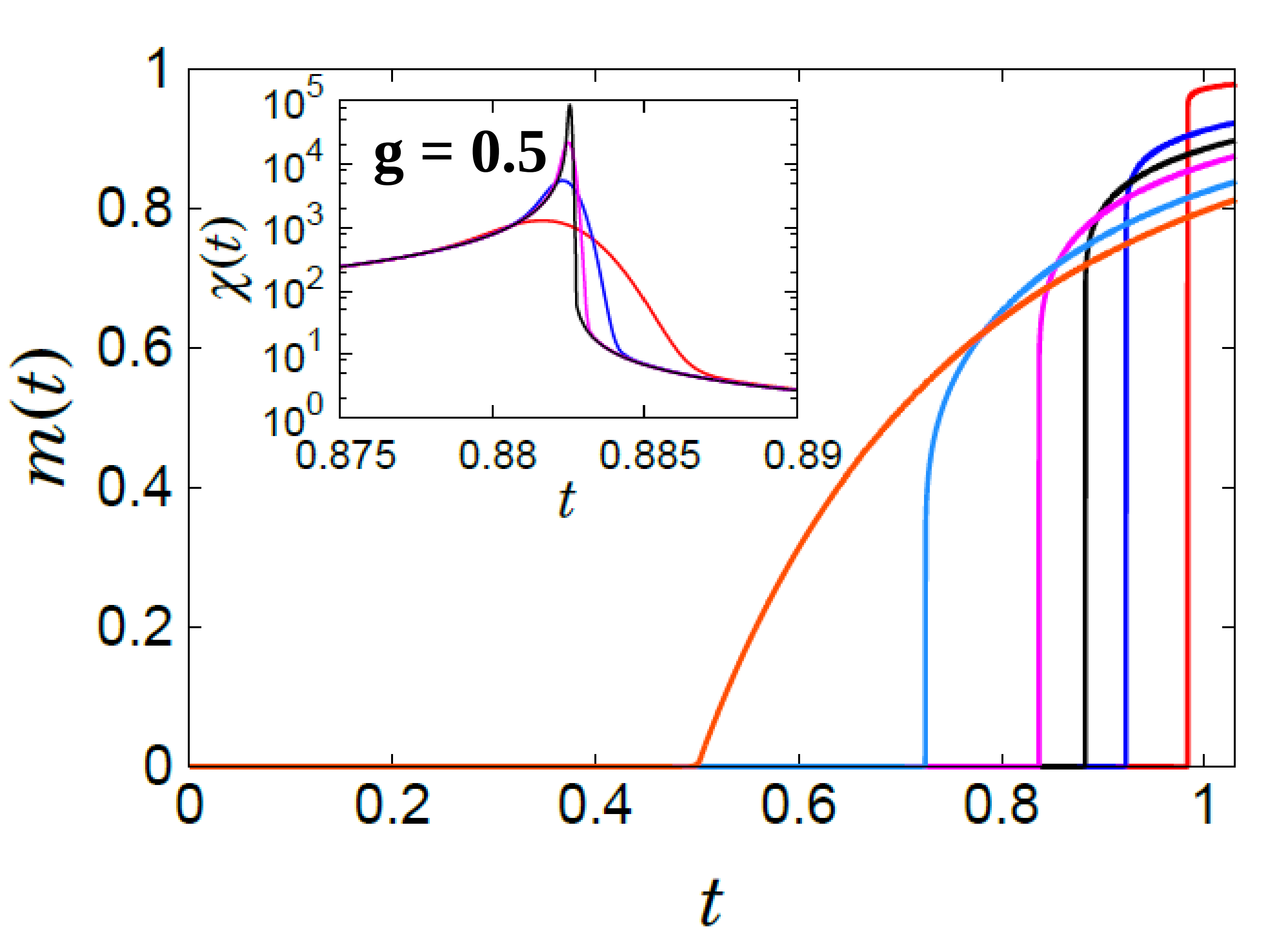
Connecting the Disconnected

- Start with *N* isolated individuals.
- R is the subset of sites belonging to the k clusters following

$$N_{k-1}(t) < [gN] \leq N_k(t) \quad \text{with} \quad N_k(t) = \sum_{l=1}^{k} s_l(t)$$

- At each step select uniformly at random one node from R and the other from the entire system.

Connecting the Disconnected



Hybrid Transition

***g* = 1, 0.8, 0.6, 0.5, 0.4, 0.2 ; *N* = 4096**

Connecting the Disconnected

Hybrid Transition

$$m(t) = \begin{cases} 0 & \text{for} \quad t < t_c \\ m_0 + r(t - t_c)^\beta & \text{for} \quad t \geq t_c \end{cases}$$

g	τ^*	τ
0.1	2.012	2.03 ± 0.04
0.2	2.061	2.08 ± 0.04
0.3	2.111	2.12 ± 0.04
0.4	2.155	2.16 ± 0.04
0.5	2.194	2.18 ± 0.04
0.6	2.231	2.20 ± 0.04
0.7	2.268	2.22 ± 0.04
0.8	2.310	2.25 ± 0.04
0.9	2.364	2.28 ± 0.04

In mean-field the cluster size exponent

$2 < \tau < 2.5$

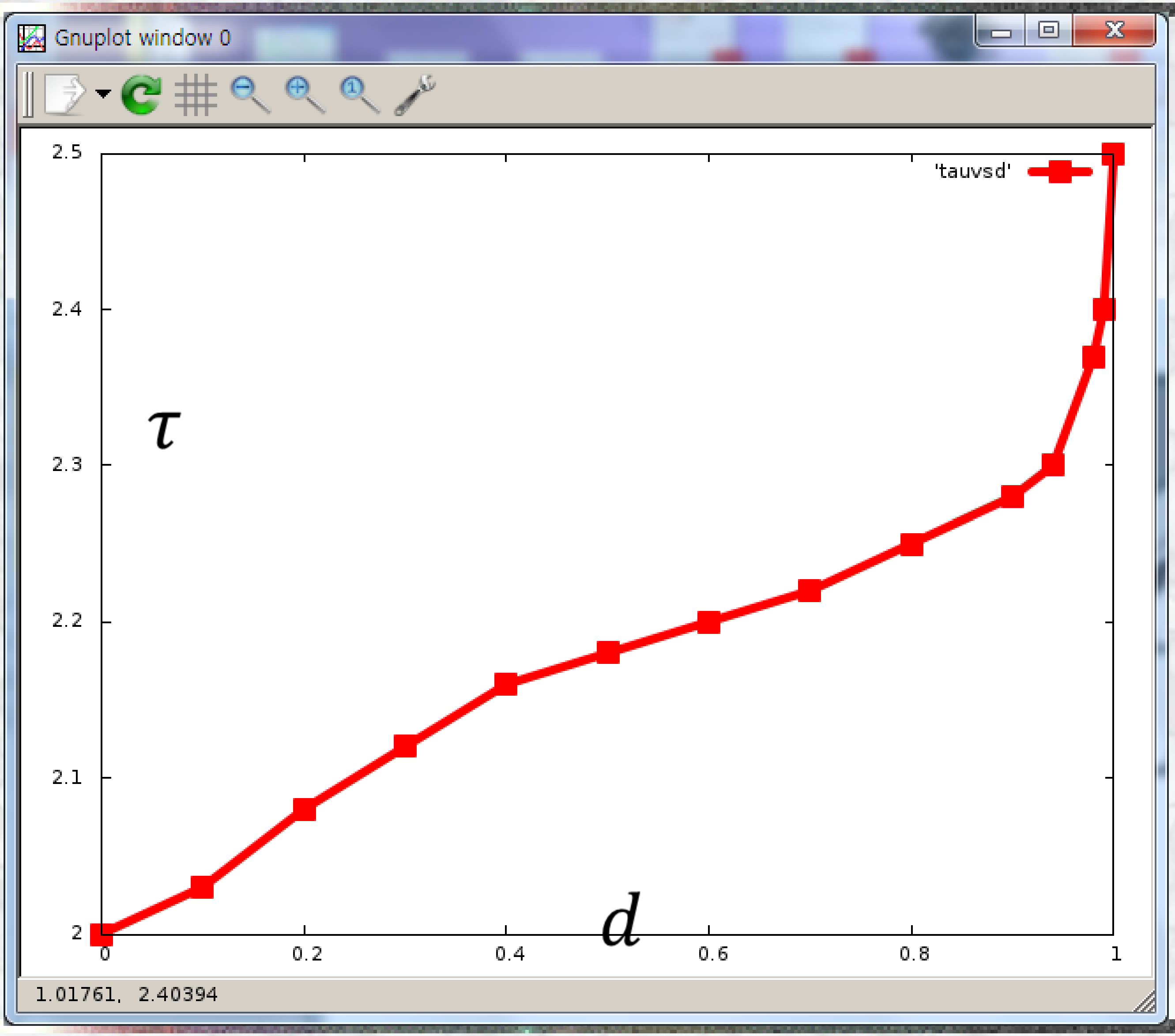
varies continuously with *g* as:

$$\frac{\varsigma(\tau)}{\varsigma(\tau-1)} = \frac{1}{g} - \frac{1}{g+1} \ln\left(\varsigma(\tau-1) \left(\frac{g+1}{2} \right)^{-\left(1+\frac{1}{g}\right)} \right)$$

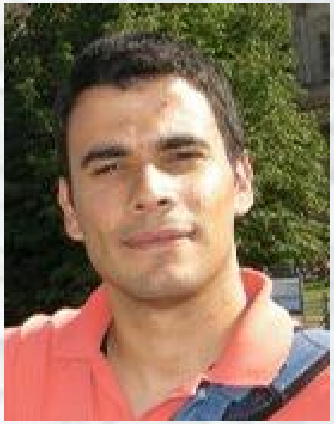
Y.S. Cho, J.S. Lee, H.J.H., B. Kahng, preprint 2015

Connecting the Disconnected

d	τ
0.1	2.03 ± 0.04
0.2	2.08 ± 0.04
0.3	2.12 ± 0.04
0.4	2.16 ± 0.04
0.5	2.18 ± 0.04
0.6	2.2 ± 0.04
0.7	2.22 ± 0.04
0.8	2.25 ± 0.04
0.9	2.28 ± 0.04
0.94	2.3 ± 0.04
0.98	2.37 ± 0.04
0.99	2.4 ± 0.04



Epidemy with Global Budget



Dirk Helbing **Lucas Böttcher** **Olivia Wooley-Meza** **Nuno Araújo**

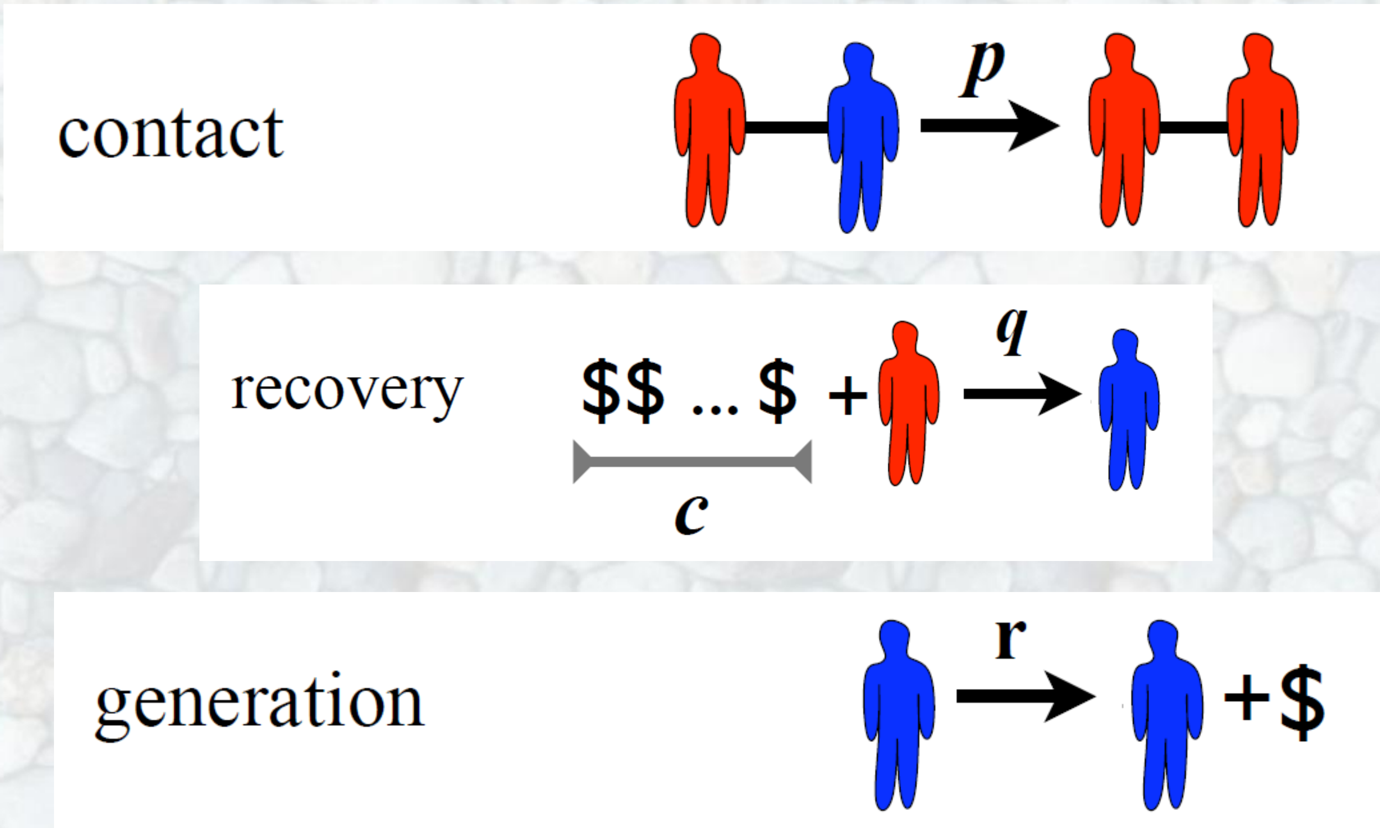
Endogenous resource constraints trigger explosive pandemics

L. Böttcher, O. Wooley-Meza, N.A.M. Araújo, H.J.H., D. Helbing

preprint

Epidemy with Global Budget

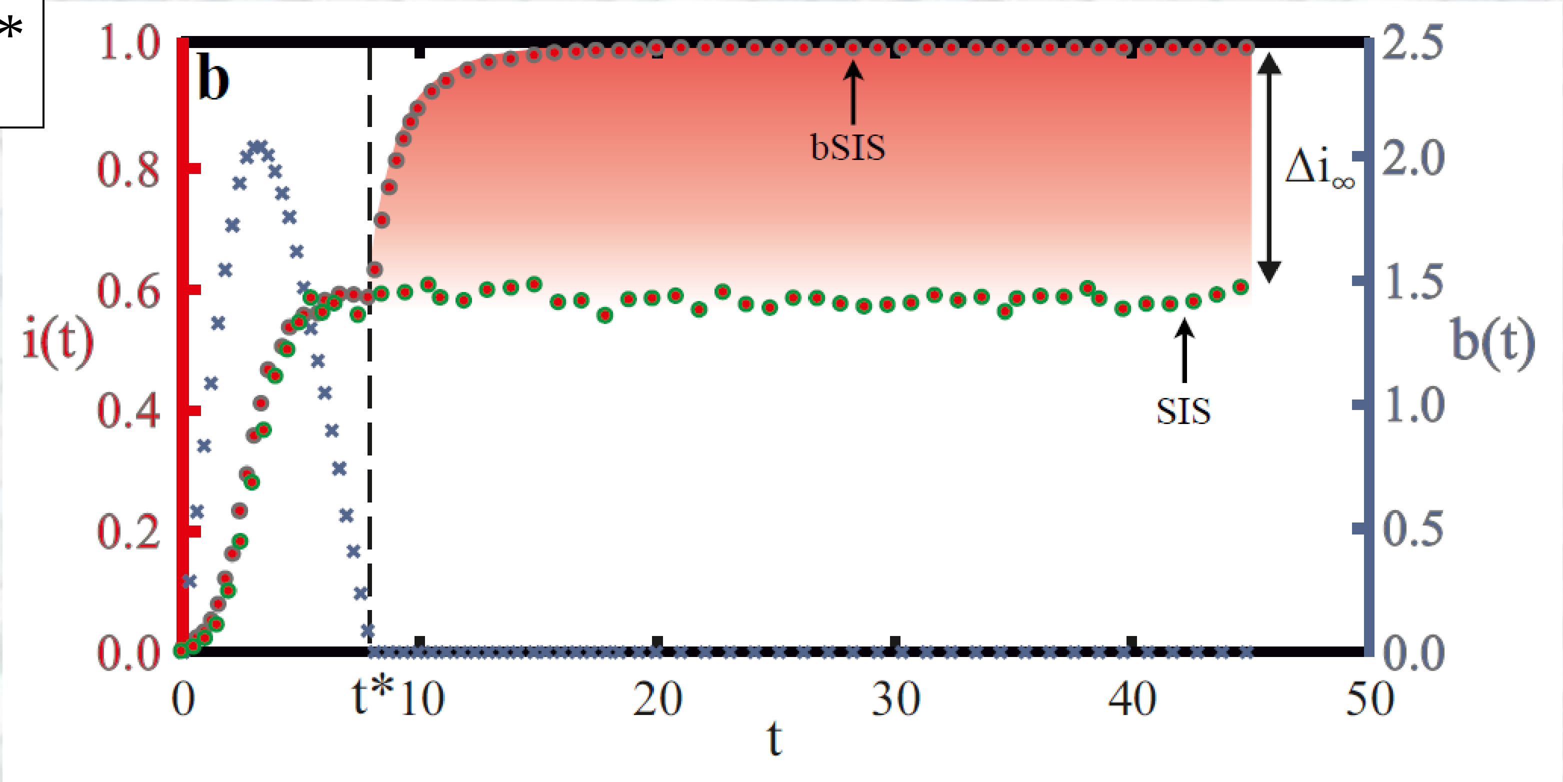
Budget-constrained Susceptible-Infected-Susceptible (bSIS) model



Epidemy with Global Budget

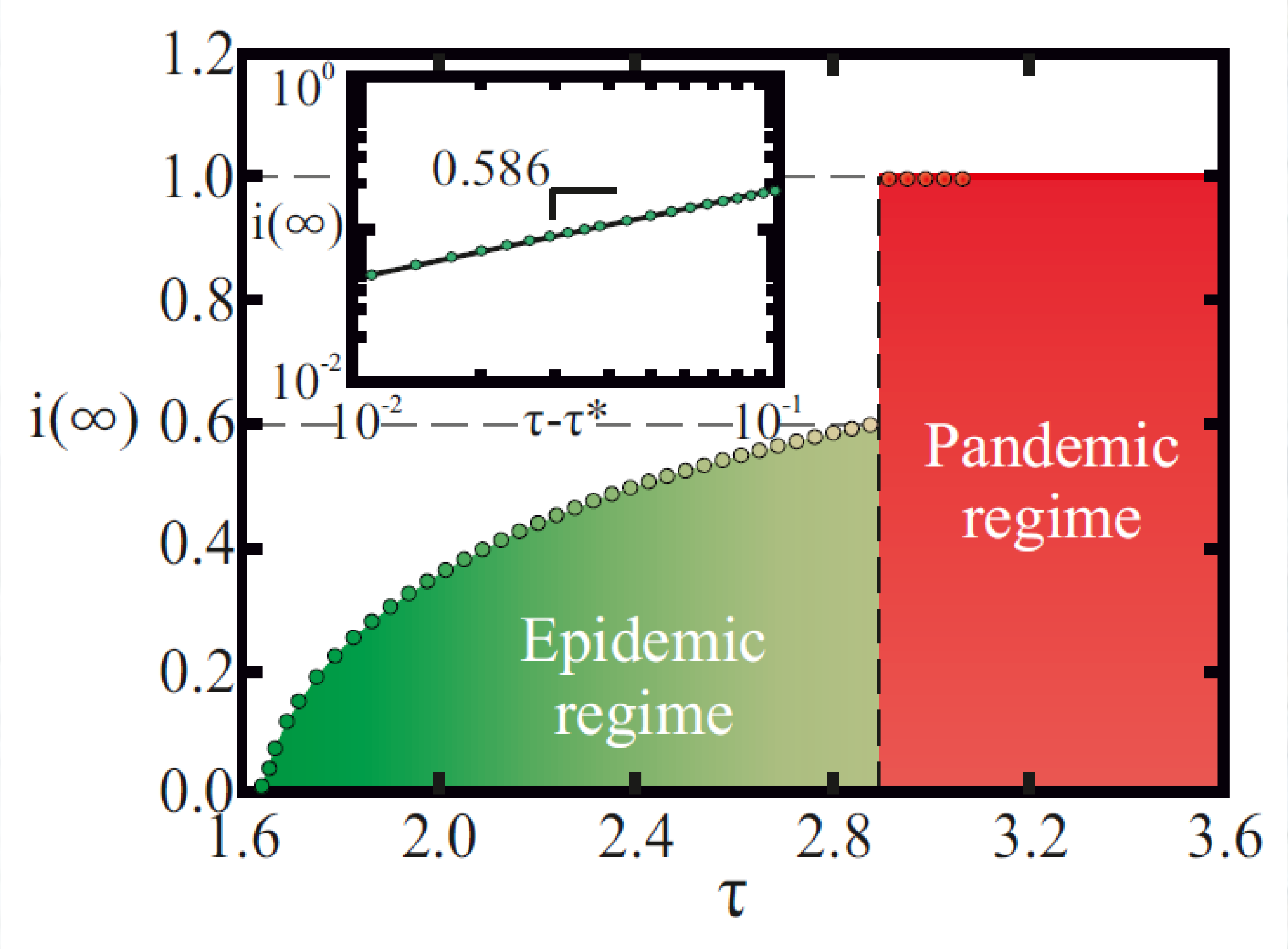
Time evolution in the epidemic regime:

$\tau > \tau^*$

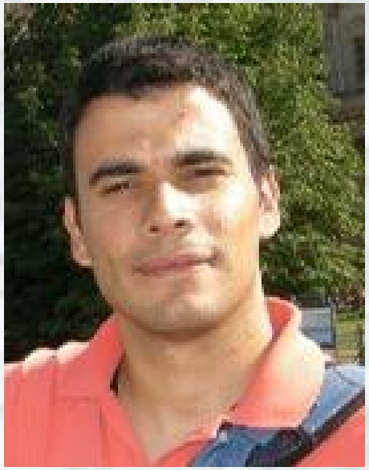


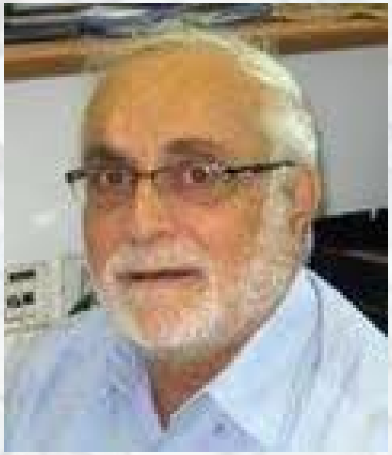
Epidemy with Global Budget

Square lattice



Coupled Networks





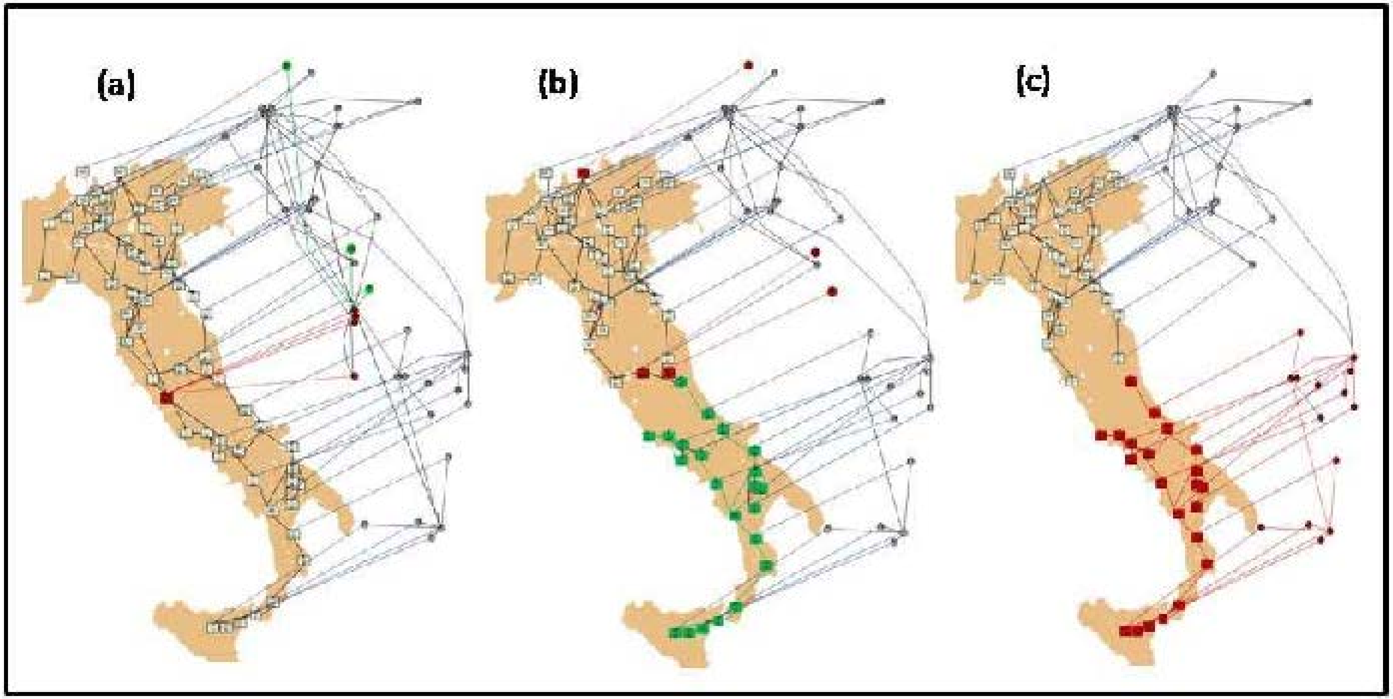
Nuno Araújo **Christian Schneider** **Shlomo Havlin**

C. Schneider, N. Yazdani, N. Araújo, S. Havlin, HJH, Sci. Rep. 3, 1969 (2013)

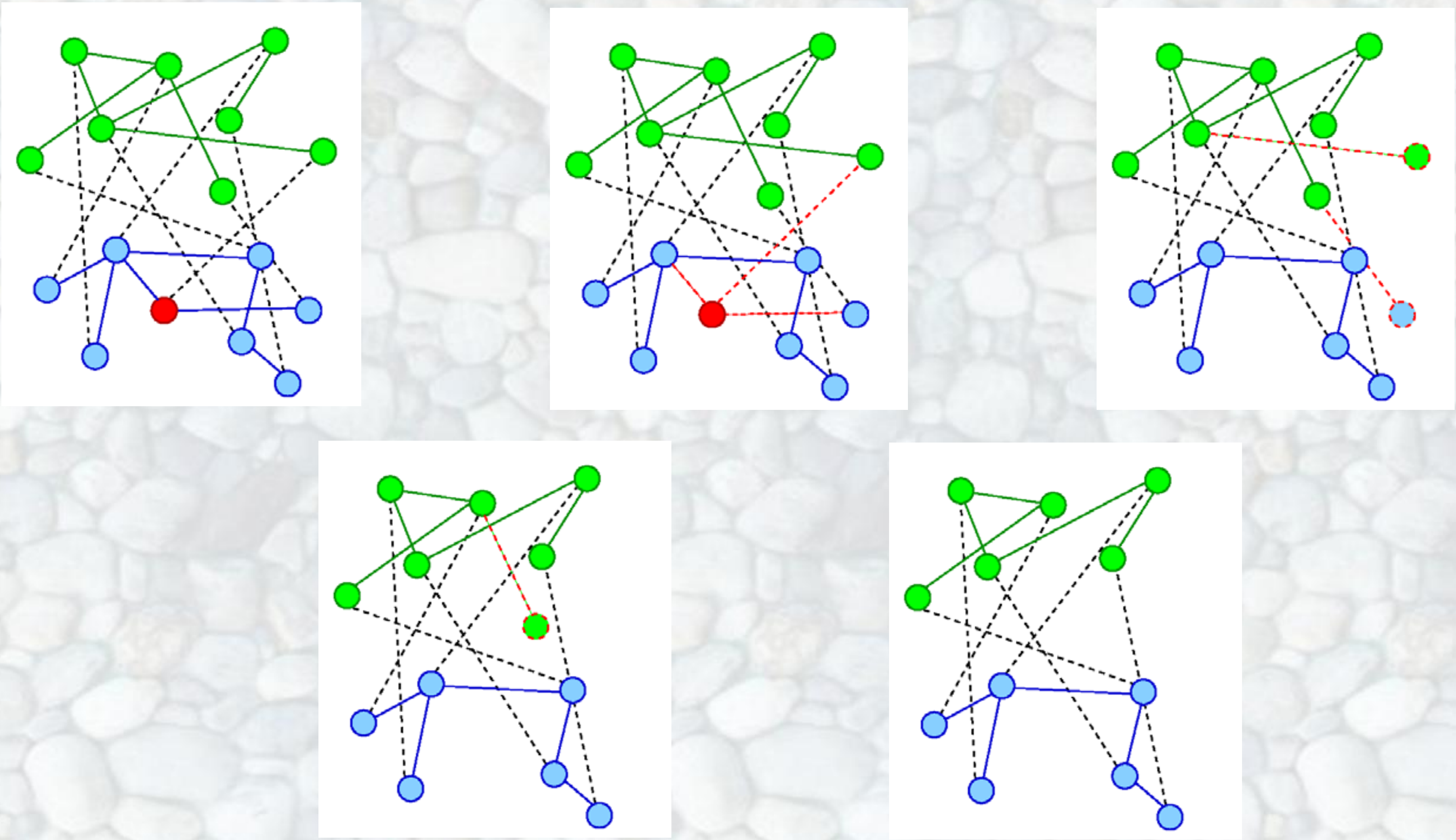
Coupled Networks

C. Schneider, N. Yazdani, N. Araújo, S. Havlin, HJH, Sci. Rep. 3, 1969 (2013)

The 2003 blackout in Italy and Switzerland

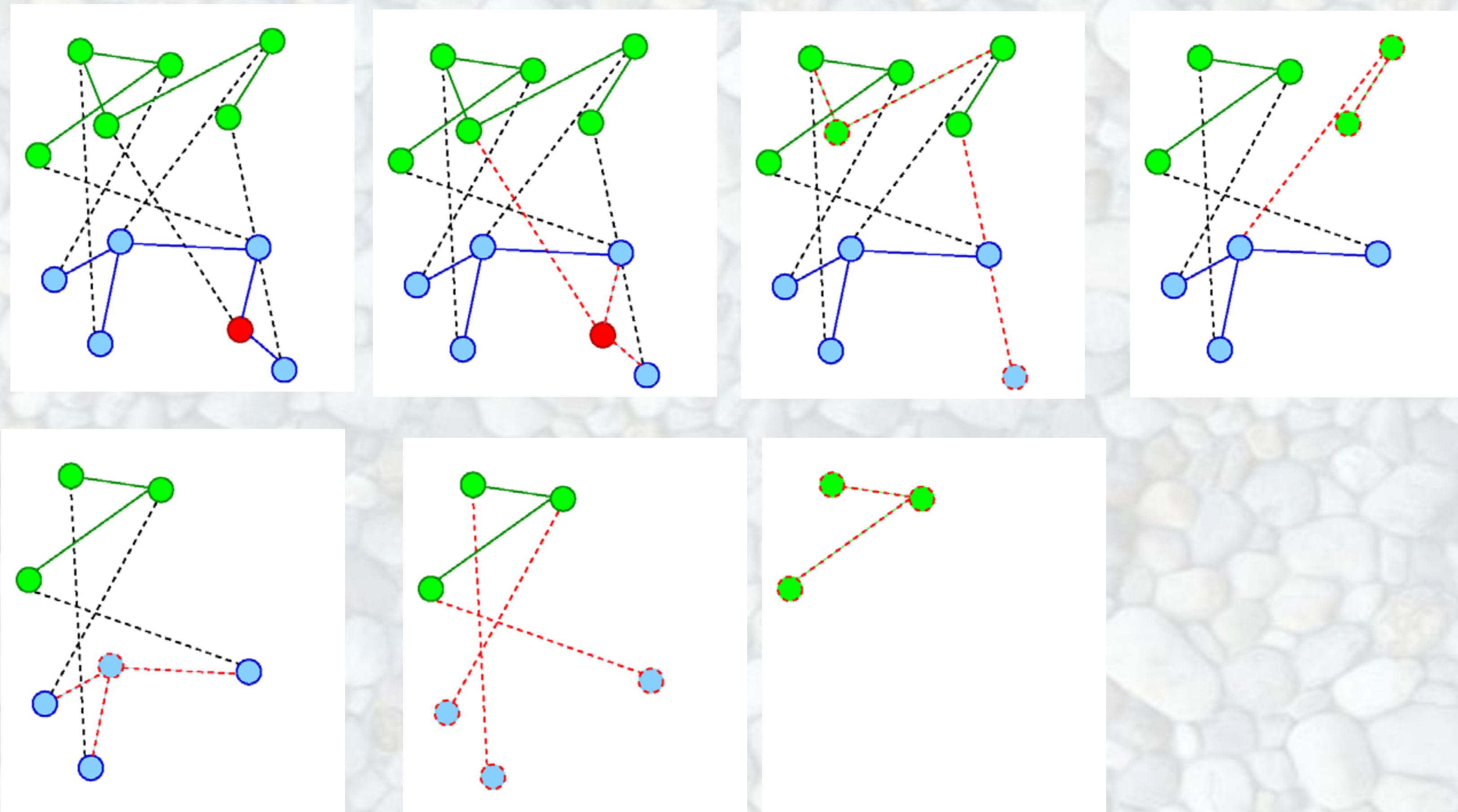


Collapse of Coupled Networks



S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin. *Nature* 464, 1025 (2010)

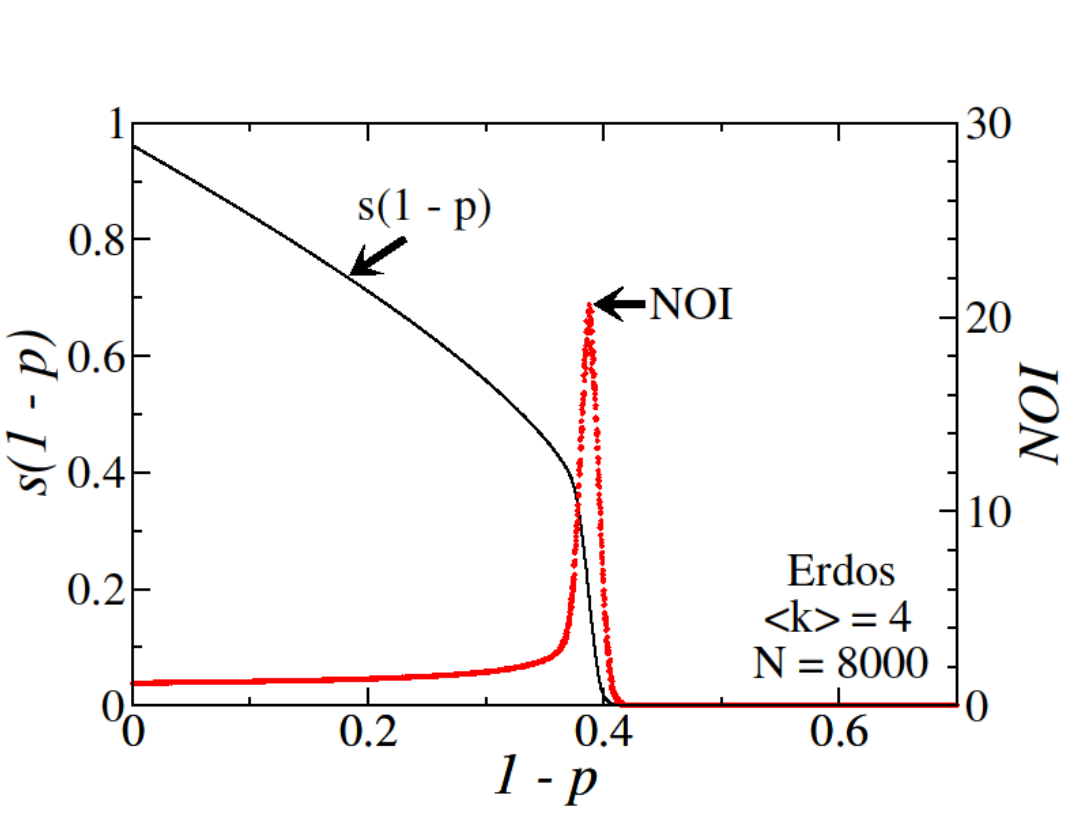
Collapse of Coupled Networks



S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin. *Nature* 464, 1025 (2010)

Collapse of Coupled Networks

Largest cluster



Number of iterations

Fraction of attacked nodes

Summary

Three Ways of creating Jump in P_∞

- **Compress the *p*-axis (e.g. by culling)**
- **Suppress the formation of a spanning cluster**
- **Increase the formation of internal bonds**

The Product Rule does the two last ones but not strongly enough.